EXPOSITIONES
MATHEMATICAE

Expo. Math. 20 (2002): 59–93
© Urban & Fischer Verlag
www.urbanfischer.de/journals/expomath

A New Look at Smoothness

Michael Maltenfort

Department of Mathematics and Statistics, Loyola University Chicago,
6525 N. Sheridan Road, Chicago, IL 60626, USA

ABSTRACT. We reconcile various authors' different definitions of smoothness. Also we give direct proofs and some new formulations of standard results about smoothness. Often hypotheses are weakened, and many examples are given to show the necessity of our hypotheses. Finally, we correct errors published in two standard references [EGA, Mu].

1. INTRODUCTION

A classical idea in analysis is to use the Jacobian matrix to distinguish "nice" points from ones which are "not nice." For instance, a continuously differentiable map is locally invertible wherever its Jacobian matrix has an inverse. Another result is that if f is a continuously differentiable function from an open set of $\mathbb{R}^s$ into $\mathbb{R}^{s-d}$, then the level set $f^{-1}(0)$ is a manifold of dimension d, provided that the Jacobian matrix has rank $s-d$ at every point of $f^{-1}(0)$ [Spv, Theorems 2-11 and 5-1]. This second result is similar to the following definition, which Zariski states in [Z3]. We have updated the definition to a scheme-theoretic framework.

Definition 1.1. *For k a field, suppose $B = k[X_1, \ldots, X_s]/(F_1, \ldots, F_t)$ is a domain. We say a point of* Spec B*, given by the prime ideal $\mathfrak{P} \subseteq B$, is an* absolutely simple point *if the Jacobian matrix $\left(\frac{\partial F_j}{\partial X_i}\right)$ has rank $s - \dim B$ when its entries are mapped to $B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}$.*

In the case where $\mathfrak{P}$ is maximal and thus defines a closed point, Zariski calls this definition the "classical and time-honored definition of simple points," even though in the paper he uses the term "absolutely simple" rather than "simple." When he later defines absolutely simple subvarieties, it corresponds, in our updated definition, to the generic point of the subvariety being absolutely simple [Z3, pp. 3-4 and §10.1, Theorem 13].

Previously, in [Z1] and [Z2], Zariski had attempted to improve this definition by eliminating any reference to the ambient space Spec $k[X_1, \ldots, X_n]$. In the earlier papers he showed that when k has characteristic zero, $\mathfrak{P}$ defines an absolutely simple point if and only if $B_{\mathfrak{P}}$ is a regular local ring; then in [Z3, §7.2, Theorem 7 and Corollary] he extends this result to allow k to be any perfect field. (We adopt modern terminology here because Zariski, confusingly, uses "simple" for regular.) It seemed that regularity might replace the notion of absolutely simple points. However, over imperfect fields, although an absolutely simple point has a regular local ring, regularity does not guarantee the point is absolutely simple. Further, regularity lacked some basic desirable properties, such as being stable under base change. So regularity did not replace the notion of absolutely simple points. In fact Mumford said that the equivalence of absolutely simple points and regularity over perfect fields k, "has historically been rather a red herring" [Mu, p. 242].

In the seminal work *Éléments de Géométrie Algébrique*, Grothendieck completely reformulated the subject with the following definition.

Definition 1.2 ([EGA, 0_{IV}.19.3.1]). *Let R be a topological ring and B a topological R-algebra. We say B is a* formally smooth *R-algebra if, for every discrete R-algebra C and every nilpotent*

E-mail address: maltenfort@yahoo.com

0732-0869/02/20/1-59 $15.00/0

ideal $J \subseteq C$, we have that every continuous R-algebra map $u : B \to C/J$ factors as $B \xrightarrow{v} C \xrightarrow{\phi} C/J$, where v is some continuous R-algebra map and ϕ is the canonical map.

The required factorization can be conveniently pictured as the existence of the indicated map in the following commutative diagram.

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dr, dashed, "v"] & C/J \\ R \arrow[u] \arrow[r] & C \arrow[u, "\phi"'] \end{tikzcd}$$

By easy induction, it can be seen that the condition that J is nilpotent can be replaced by the condition that $J^2 = 0$. Our definition of quasi-smooth, which we will give in Section 2, will incorporate this change and omit the topological conditions. That is, the ring map $R \to B$ will be quasi-smooth if and only if B is a formally smooth R-algebra when R and B are each given the discrete topology.

The new definition of formal smoothness does capture the notion of absolutely simple. Indeed, using the notation of Definition 1.1, the point defined by $\mathfrak{P}$ is absolutely simple if and only if $B_{\mathfrak{P}}$ is a formally smooth k-algebra, for k and $B_{\mathfrak{P}}$ each being given the discrete topology. That is, absolute smoothness of a point is equivalent to formal smoothness of the local k-algebra at that point. We will show this in Corollary 7.4. But we can discuss formal smoothness of any algebra, not just local ones; this is one way in which formally smooth algebras are more general than absolutely simple points. Indeed, even though formal smoothness of an algebra implies formal smoothness for all localizations at a prime ideal, the converse is not true, as we will see in Example 4.8. (Most applications, however, meet the mild hypothesis of Theorem 4.7, so that this issue doesn't arise.)

Unlike in the definition of absolutely simple points, formally smooth algebras can have a base ring which is not a field, and this expansion is quite significant. In particular, it has applications in arithmetic geometry, when the base ring might be $\mathbb{Z}$ or a ring of integers in a number field. The final way in which formally smooth algebras generalize absolutely simple points is by working in the category of topological rings and continuous maps, but this last aspect will not be addressed in this paper.

To turn to geometry from algebra, let us look at the definition in *Éléments de Géométrie Algébrique* for a map of schemes to be formally smooth.

Definition 1.3 ([EGA, IV.17.1.1]). *Let $f : X \to S$ be a map of schemes. We say f is* formally smooth *[resp.* formally unramified, formally étale*] if, for every affine scheme* $\operatorname{Spec} C$*, every closed subscheme* $\operatorname{Spec} C/J$ *defined by a nilpotent ideal $J \subseteq C$, and every map* $\operatorname{Spec} C \to S$*, the map* $\operatorname{Hom}_S(\operatorname{Spec} C, X) \to \operatorname{Hom}_S(\operatorname{Spec} C/J, X)$ *induced by the canonical inclusion* $\operatorname{Spec} C/J \to \operatorname{Spec} C$ *is surjective [resp. injective, bijective].*

We can interpret this definition using a diagram dual to the one above. The required condition, restated, says that the indicated map in the following commutative diagram exists [resp. is unique, exists and is unique].

$$\begin{tikzcd} X \arrow[d] & \operatorname{Spec} C/J \arrow[l] \arrow[d] \\ S & \operatorname{Spec} C \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

It is obvious that a map of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} R$ is formally smooth if and only if B is a formally smooth R-algebra, when each ring is given the discrete topology. The definition of a map of schemes to be formally smooth corresponds exactly, with the same trivial adjustment to $J^2 = 0$, to that of our quasi-smooth which we will define in Section 3. The terms formally unramified and formally étale will be discussed in Remark 3.5.

The diagram above bears a striking resemblance to the homotopy lifting property from topology, which, let us recall, is defined as follows [Spn]. Fix topological spaces E, B, and Y and a continuous map $p : E \to B$. Let $I = [0, 1]$ be the unit interval and let $i_t : Y \to Y \times I$ be inclusion at t, that is, $y \mapsto (y, t)$, for $t = 0$ or 1. We say p has the *homotopy lifting property with respect to the space* Y if for all continuous maps $f' : Y \to E$ and $F : Y \times I \to B$ such that $pf' = Fi_0$, there exists a continuous map $F' : Y \times I \to E$ such that $F'i_0 = f'$ and $pF' = F$, as illustrated below. Here we think of F as giving a homotopy $Fi_1 \simeq Fi_0 = pf'$, and F' then lifts this to a homotopy $F'i_1 \simeq F'i_0 = f'$.

$$\begin{tikzcd} E \arrow[d, "p"'] & Y \arrow[l, "f'"'] \arrow[d, "i_0"] \\ B & Y \times I \arrow[l, "F"] \arrow[ul, dashed, "F'"'] \end{tikzcd}$$

The continuous map $p : E \to B$ is said to be a *fibration* (or a *Hurewicz fiber space*) if it has the homotopy lifting property for all spaces Y. So formal smoothness is a close analogue to the earlier concept of a topological fibration. With this interpretation, the desired map $\operatorname{Spec} C \to X$ is thought of as a lift of $\operatorname{Spec} C \to S$. This is somewhat unsatisfactory, however, especially when the base scheme S is simply a one point space $\operatorname{Spec} k$, for k a field.

For a different geometric interpretation of formal smoothness, we can think of $\operatorname{Spec} C \to X$ as an extension of $\operatorname{Spec} C/J \to X$ to a larger domain, since $\operatorname{Spec} C/J$ is a closed subscheme of $\operatorname{Spec} C$. Let us recall that if $J \subseteq C$ is a nilpotent ideal, then $\operatorname{Spec} C$ is identical as a topological space to its closed subscheme $\operatorname{Spec} C/J$, and we think of $\operatorname{Spec} C$ as a "thickening" of $\operatorname{Spec} C/J$. (In fact, both are thickened from the reduced scheme $\operatorname{Spec} C/\operatorname{nil} C$.) So a map of schemes $X \to S$ is formally smooth if every map $\operatorname{Spec} C/J \to X$ of schemes over S can be extended to a map $\operatorname{Spec} C \to X$ from such a thickening.

In the case of affine schemes $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$, we can push this interpretation further. Consider embedding $\operatorname{Spec} B$ as a closed subscheme of $\operatorname{Spec} R[X_\lambda]$ (which we think of as an affine n-space over $\operatorname{Spec} R$), say $\operatorname{Spec} B = \operatorname{Spec} R[X_\lambda]/I$. By Remark 2.6, such an embedding always exists, provided that we allow an infinite number of variables. Then by Lemma 2.5, $\operatorname{Spec} B \to \operatorname{Spec} R$ is formally smooth if and only if the indicated map in the following commutative diagram exists, where the top map is our chosen isomorphism and the right hand map is inclusion of a closed subscheme.

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d] & \operatorname{Spec} R[X_\lambda]/I \arrow[l] \arrow[d] \\ \operatorname{Spec} R & \operatorname{Spec} R[X_\lambda]/I^2 \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

Now, we think of $\operatorname{Spec} R[X_\lambda]/I^2$ as a thickening of $\operatorname{Spec} B$ in the ambient "affine n-space over $\operatorname{Spec} R$." So $\operatorname{Spec} B \to \operatorname{Spec} R$ will be formally smooth exactly when there exists a map from this thickening to $\operatorname{Spec} B$ itself such that when this map follows the inclusion of $\operatorname{Spec} B$ in its thickening, we get the identity. That is, given a map $X \to S$ of affine schemes and any closed embedding of X in an "affine n-space over S," $X \to S$ is formally smooth if and only if X is a retract of a particular thickening in this embedding, that is, the inclusion map has a left inverse.

Grothendieck's algebraic and geometric definitions of formal smoothness replaced the concept of absolutely simple points, and this new concept has become one important standard by which maps of rings or schemes are judged "nice." A large problem, however, is that many authors use quite different definitions of smoothness and fail to show equivalence to other definitions. Another difficulty is that authors have lacked uniformity in the terminology for smoothness, somewhat similar to the confusion we have already seen among the terms regular, simple, and absolutely simple. (Because they seem the least ambiguous, we use Swan's terms [Sw] quasi-smooth, essentially smooth, and smooth, which we introduce in Sections 2, 3, 7, and 8.) One aim of this paper, which we will accomplish in Section 8, is to prove equivalence of various definitions and to clarify the terminology in a variety of standard resources [EGA, AK, H, Mat1, Mat2, Mu, F, Sw]. Additionally in this paper we will generalize results, weaken hypotheses, provide numerous examples, and correct two previously published errors.

In Section 2 and Section 3, we will give general results on quasi-smoothness of maps of rings and maps of schemes, respectively. In each following section, we get progressively stronger results by restricting the class of maps which we consider.

In Section 4, we introduce a fairly weak restriction on the module or sheaf of differentials. At this level of generality we prove most results on the local nature of quasi-smoothness. In Section 5, we reduce to quasi-smoothness of fibers using a more restrictive condition which we call conormally finite. This condition first appeared, unnamed, in [EGA, 0_{IV}.22.6.4], and it is only at this level that Grothendieck proves one way in which smoothness is local [EGA, 0_{IV}.22.6.6]. He proves another result on the local nature of quasi-smoothness in [EGA, IV.17.1.6], but uses an unstated hypothesis which is stronger than our condition in Section 4. We discuss this in detail at the end of Section 4, where we also pose several open questions.

In Section 6, we prove the Jacobian Criterion and calculate the quasi-smooth locus. Section 7 deals with essentially finitely presented maps, and we connect essential smoothness to flatness and regular sequences. For Noetherian rings, it is also at this level of generality that we relate essential smoothness to dimension and regular rings. Finally, in addition to clarifying definitions and terminology, in Section 8 we examine finitely presented maps and relate smoothness to equidimensional fibers. This section concludes with a correction of a theorem in Mumford's *Red Book of Varieties and Schemes* [Mu].

We assume that all rings are commutative with unit. Unless stated, we do not assume that rings, local or otherwise, are Noetherian, that schemes are Noetherian or separated, or that points of a scheme are closed. If $\mathfrak{p} \subseteq R$ is a prime ideal, $\kappa(\mathfrak{p})$ denotes the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Similarly, for X a scheme and $x \in X$, $k(x)$ denotes the residue field of the local ring at x. We say a map of local rings $R \to B$ is local if the maximal ideal of R is the preimage of the maximal ideal of B. For an affine scheme $\operatorname{Spec} A$, a distinguished open set means an open set of the form $\operatorname{Spec} A_a$. We say a map of modules $f : M \to N$ is a split injection if there is a module map $g : N \to M$ such that $gf = 1_M$, i.e. f is the inclusion of a direct summand.

When we use the notation $R[X_\lambda]$, we mean a polynomial ring in a possibly infinite number of variables, whereas $R[X_i]$ means only finitely many variables. The notation $\oplus_\lambda B\, dX_\lambda$ means the free B-module with the set of symbols $\{dX_\lambda\}$ as basis. When we say a map is a lift, it must be a homomorphism of rings or schemes which makes the appropriate diagram commute.

Acknowledgements

The author wishes to thank Ian Robertson, David Schmitz, M. Pavaman Murthy, and Madhav Nori for many helpful conversations.

2. Quasi-Smooth Maps of Rings

Definition 2.1. *We say a map of rings $R \to B$ is* quasi-smooth *(or B is a* quasi-smooth R-algebra*) if for every R-algebra C, ideal $J \subseteq C$ with $J^2 = 0$, and every R-algebra map $B \to C/J$, there exists a lift to an R-algebra map $B \to C$. That is, $R \to B$ is quasi-smooth if the indicated lift exists in all commutative diagrams such as the following, when $J^2 = 0$.*

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dashed] & C/J \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

As we have mentioned, this definition, and those for "smooth" and "essentially smooth" which we will give in Section 7, follow the terminology of Swan [Sw]; for a comparison to other authors' definitions of smoothness, see Section 8. Some results such as the Jacobi–Zariski Sequence (which generalizes the "fundamental exact sequences") are not used in this paper, and will not be proved here, since they are presented clearly in [Sw].

We begin with Proposition 2.2, which lists several standard facts that follow immediately from the above definition. For (4), use Lemma 2.3. The proofs are left to the reader, or see [EGA], [Mat1], or [Mat2]. These facts will be used without further reference.

Proposition 2.2.

(1) A polynomial extension $R \to R[X_\lambda]$ is quasi-smooth.
(2) The composition of quasi-smooth maps is quasi-smooth.
(3) If $R \to B$ is quasi-smooth and $R \to R'$ is any map, then the base change $R' \to B \otimes_R R'$ is quasi-smooth.
(4) For $S \subseteq R$ a multiplicative set, $R \to R_S$ is quasi-smooth.

Lemma 2.3. *If C is a ring and $J \subseteq C$ an ideal which lies inside the Jacobson radical, then an element of C is a unit if and only if its image in C/J is a unit.*

Proposition 2.4. *Suppose we have a map $R \to B$ with B local. Let $\mathfrak{p} \subseteq R$ be the inverse image of the maximal ideal of B. Then $R \to B$ is quasi-smooth if and only if $R_\mathfrak{p} \to B$ is quasi-smooth.*

Proof. "If" follows from Proposition 2.2(4),(2). "Only if" follows directly from the definition of quasi-smooth, since an R-algebra map of $R_\mathfrak{p}$-algebras is automatically an $R_\mathfrak{p}$-algebra map. □

The following lemma allows us to prove a map is quasi-smooth by checking that only one particular map has a lift.

Lemma 2.5. *Suppose we have a map $R \to A$ and $I \subseteq A$ an ideal. If $R \to A/I$ is quasi-smooth, then there exists a lift as indicated in the following diagram, where the top map is the identity. Conversely, if such a lift exists and $R \to A$ is quasi-smooth, then $R \to A/I$ is quasi-smooth.*

$$\begin{tikzcd} A/I \arrow[r] \arrow[dr, dashed] & A/I \\ R \arrow[u] \arrow[r] & A/I^2 \arrow[u] \end{tikzcd}$$

Remark 2.6. For a ring map $R \to B$, we can find a presentation $B = A/I$ where $R \to A$ is quasi-smooth, which is what we need in Lemma 2.5. Indeed, let $A = R[X_\lambda]_{\lambda \in B}$ and let $I \subseteq A$ be the kernel of the R-algebra surjection $A \twoheadrightarrow B$ which sends $X_\lambda \mapsto \lambda$.

Proof. The first statement follows directly from the definition of quasi-smooth. So suppose $R \to A$ is quasi-smooth and we have a lift $A/I \to A/I^2$ as in the diagram above. Let C be an R-algebra, $J \subseteq C$ an ideal with $J^2 = 0$, and $A/I \to C/J$ an R-algebra map. Since $R \to A$ is quasi-smooth, find an R-algebra lift $A \to C$ of the composition $A \to A/I \to C/J$. The lift must send I into J, and so we get an induced $A/I^2 \to C/J^2 = C$ which makes the following diagram commute.

$$
\begin{tikzcd}
A/I \arrow[rr] \arrow[dr, bend right] & & C/J \\
& A/I^2 \arrow[ul] \arrow[ddr, dashed] & \\
& A \arrow[u] \arrow[dr, dashed] & \\
R \arrow[uuu] \arrow[ur] \arrow[rr] & & C \arrow[uuu]
\end{tikzcd}
$$

The composition $A/I \to A/I^2 \to C$ is the desired lift of $A/I \to C/J$, showing $R \to A/I$ is quasi-smooth. □

Example 2.7. An R-module M is projective iff $R \to \operatorname{Sym}_R^\bullet M$ is quasi-smooth.

Let $M = F/N$ with F a free R-module, and let $I \subseteq \operatorname{Sym}_R^\bullet F$ be the ideal generated by $N \subseteq \operatorname{Sym}_R^1 F$, so that $(\operatorname{Sym}_R^\bullet F)/I = \operatorname{Sym}_R^\bullet M$. Being a polynomial extension, $R \to \operatorname{Sym}_R^\bullet F$ is quasi-smooth. By Lemma 2.5, $R \to \operatorname{Sym}_R^\bullet M$ is quasi-smooth iff there is an R-algebra map $\operatorname{Sym}_R^\bullet M \to (\operatorname{Sym}_R^\bullet F)/I^2$ such that the following composition equals the identity map: $\operatorname{Sym}_R^\bullet M \to (\operatorname{Sym}_R^\bullet F)/I^2 \to (\operatorname{Sym}_R^\bullet F)/I = \operatorname{Sym}_R^\bullet M$.

If such a lift exists, then using inclusion of and projection to degree 1, we get the following R-module map: $M \hookrightarrow \operatorname{Sym}_R^\bullet M \to (\operatorname{Sym}_R^\bullet F)/I^2 \twoheadrightarrow F$. This map gives us a splitting to show that $F \cong M \oplus N$; so M is projective. Conversely, if M is projective, such a splitting $M \hookrightarrow F$ induces the desired lift: $\operatorname{Sym}_R^\bullet M \to \operatorname{Sym}_R^\bullet F \twoheadrightarrow (\operatorname{Sym}_R^\bullet F)/I^2$.

Although quasi-smoothness looks for the existence of a lifting map, it will be useful to look at all such lifting maps, for which we have the following result.

Proposition 2.8. *Consider the following diagram of ring maps, where $J \subseteq C$ is an ideal such that $J^2 = 0$.*

$$
\begin{tikzcd}
A \arrow[r] \arrow[dr, dashed] & C/J \\
R \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}
$$

Suppose a lift $\theta : A \to C$ exists as indicated above, and let $f : A \to C$ be a map of sets. Then $\theta - f$ is another lift of the diagram above if and only if we have all of the following conditions.

(1) For all $a \in A$, $f(a) \in J$.
(2) If $a \in A$ is in the image of R, $f(a) = 0$.
(3) For all $a_1, a_2 \in A$, $f(a_1 + a_2) = f(a_1) + f(a_2)$.
(4) For all $a_1, a_2 \in A$, $f(a_1 a_2) = a_1 f(a_2) + a_2 f(a_1)$.

In (4), we use the A-module structure on the C/J-module J which is induced by the given map $A \to C/J$.

Proof. As maps of sets only, $\theta - f$ makes the above diagram commute iff (1) and (2) hold; note that (2) automatically implies $(\theta - f)(1) = 1$. Also $\theta - f$ will be a homomorphism of abelian groups iff (3) holds. So we see that $\theta - f$ is a lift as desired iff we have (1), (2), (3), and $(\theta - f)(a_1a_2) = (\theta - f)(a_1)(\theta - f)(a_2)$. But this last condition is equivalent to $-f(a_1a_2) = -\theta(a_1)f(a_2) - \theta(a_2)f(a_1) + f(a_1)f(a_2)$. Since $J^2 = 0$, $f(a_1)f(a_2) = 0$; and θ induces the same A-module structure on J as described in the statement of the proposition. Thus, this last condition is equivalent to (4). □

A map f satisfying conditions (2), (3), and (4) in Proposition 2.8 is known as an R-derivation of A. These conditions motivate the following definition.

Definition 2.9. *Let $R \to A$ be map of rings. We define $\Omega_{A/R}$, the* module of differentials, *to be the A-module generated by the set of symbols $\{da : a \in A\}$ with the following relations.*

- *If $a \in A$ is in the image of R, $da = 0$.*
- *For all $a_1, a_2 \in A$, $d(a_1 + a_2) = da_1 + da_2$.*
- *For all $a_1, a_2 \in A$, $d(a_1a_2) = a_1\, da_2 + a_2\, da_1$.*

Remarks 2.10.

(1) For $r \in R$ and $a \in A$, $d(ra) = r\,da$; for $n \geq 2$, $d(a^n) = na^{n-1}\,da$; and if $b \in A$ is a unit, $d\frac{a}{b} = \frac{b\,da - a\,db}{b^2}$.
(2) If $a \in A$ is idempotent, then in $\Omega_{A/R}$, $da = 0$. Indeed, use (1) for the third equality: $0 = d(2a - 3a + a) = d(2a^3 - 3a^2 + a) = (6a^2 - 6a + 1)da = (6a - 6a + 1)da = da$. So if $\{a_\lambda\} \subseteq A$ is a set of idempotents, then $\Omega_{R[a_\lambda]/R} = 0$ and $\Omega_{A/R[a_\lambda]} = \Omega_{A/R}$.
(3) In the notation of Proposition 2.8, there is a natural bijective correspondence between $\mathrm{Hom}_A(\Omega_{A/R}, J) = \mathrm{Hom}_{C/J}(\Omega_{A/R} \otimes_A C/J, J)$ and those f such that $\theta - f$ is a lift. In such a case, it will be convenient to use f to indicate both a map $\Omega_{A/R} \otimes_A C/J \to J$ and its corresponding map of sets $A \to J$.

Example 2.11. $\Omega_{R[X_\lambda]_S/R}$ is a free $R[X_\lambda]_S$-module with basis $\{dX_\lambda\}$.

For $S \subseteq R[X_\lambda]$ a multiplicative set, let $A = R[X_\lambda]_S$. Of course, we have a natural A-module map $\oplus_\lambda A\,dX_\lambda \to \Omega_{A/R}$ given by $dX_\lambda \mapsto dX_\lambda$. To find an inverse, use the relations defining $\Omega_{A/R}$ and those in Remark 2.10(1) to conclude that in $\Omega_{A/R}$, $dF = \sum_\lambda \frac{\partial F}{\partial X_\lambda}\,dX_\lambda$, for $F \in A$. (Induct on the complexity of F.) The inverse to our map above will be the A-module map $\Omega_{A/R} \to \oplus_\lambda A\,dX_\lambda$ defined by $dF \mapsto \sum_\lambda \frac{\partial F}{\partial X_\lambda}\,dX_\lambda$. This map is well defined since it is zero on the relations of $\Omega_{A/R}$, using properties of partial derivatives.

Example 2.12. For $A' = A \otimes_R R'$, $\Omega_{A'/R'} = \Omega_{A/R} \otimes_A A'$.

For A and R' any R-algebras, by mapping $d(a \otimes 1)$ and $(da) \otimes 1$ to one another, we define mutually inverse A'-module maps showing $\Omega_{A'/R'} = \Omega_{A/R} \otimes_R R'$, and the latter module equals $\Omega_{A/R} \otimes_A A'$.

Example 2.13. $\Omega_{A_S/R} = (\Omega_{A/R})_S$.

Let A be an R-algebra and $S \subseteq A$ a multiplicative set. Mapping $da \mapsto d\frac{a}{1}$ gives an A_S-module map $(\Omega_{A/R})_S \to \Omega_{A_S/R}$. From the final equation of Remark 2.10(1), the inverse to this map should be the A_S-module map given by $d\frac{a}{s} \mapsto \frac{s\,da - a\,ds}{s^2}$, and we only need to verify this map is well defined. On the generators $d\frac{a}{s}$ of $\Omega_{A_S/R}$ it is well defined, because $d\frac{as'}{ss'} \mapsto \frac{ss'\,d(as') - as'\,d(ss')}{(ss')^2} = \frac{s\,da - a\,ds}{s^2}$, which is the image of $d\frac{a}{s}$. As for the relations on $\Omega_{A_S/R}$, clearly this map vanishes on $d\frac{a}{1}$ for a in the image of R. To verify the other relations, we may write two arbitrary elements of A_S with equal denominators. Then $d\left(\frac{a}{s} + \frac{b}{s}\right) \mapsto \frac{s\,d(a+b) - (a+b)\,ds}{s^2}$

which equals the image of $d\frac{a}{s} + d\frac{b}{s}$. Finally, $d\left(\frac{a}{s}\frac{b}{s}\right) \mapsto \frac{s^2\, d(ab) - ab\, d(s^2)}{s^4} = \frac{as\, db + bs\, da - 2ab\, ds}{s^3} = \frac{a}{s}\frac{s\, db - b\, ds}{s^2} + \frac{b}{s}\frac{s\, da - a\, ds}{s^2}$, which is the image of $\frac{a}{s}\, d\frac{b}{s} + \frac{b}{s}\, d\frac{a}{s}$.

Now we introduce one of our key tools, the differential map, the importance of which will be seen in Theorem 2.17 and Corollary 2.18.

Definition 2.14. *For a map of rings $R \to A$ and an ideal $I \subseteq A$, we have a natural A/I-module map $d^I_{A/R} : I/I^2 \to \Omega_{A/R} \otimes_A A/I$ induced by $a \mapsto da$. We call $d^I_{A/R}$ the* differential map. *Note that the cokernel of $d^I_{A/R}$ is $\Omega_{(A/I)/R}$.*

Remarks 2.15. Let R, A, and I be as above, and $B = A/I$.

(1) Let R' be an R-algebra, $A' = A \otimes_R R'$, $I' = IA'$, and $B' = A'/I'$. Using Example 2.12, $d^I_{A/R} \otimes_B B'$ is equal to the canonical surjection $I/I^2 \otimes_B B' \twoheadrightarrow I'/(I')^2$ followed by $d^{I'}_{A'/R'}$.

(2) Let $T \subseteq A$ be a multiplicative set with image $S \subseteq B$. Using Example 2.13, $d^I_{A/R} \otimes_B B_S = d^{I_T}_{A_T/R}$.

Lemma 2.16. *Let $R \to A$ be any map and $I \subseteq A$ an ideal. Then $d^I_{A/R}$ is a split injection if and only if there exists some lift in the following diagram, where the top map is the identity.*

$$\begin{tikzcd} A/I \arrow[r] \arrow[dr, dashed] & A/I \\ R \arrow[u] \arrow[r] & A/I^2 \arrow[u] \end{tikzcd}$$

Proof. More generally we show a one-to-one correspondence between $f \in \operatorname{Hom}_{A/I}(\Omega_{A/R} \otimes_A A/I, I/I^2)$ such that $f d^I_{A/R} = 1_{I/I^2}$ and lifts in the above diagram. Indeed, such lifts correspond to lifts $A \to A/I^2$ in the following diagram such that I is mapped to zero.

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & A/I \\ R \arrow[u] \arrow[r] & A/I^2 \arrow[u] \end{tikzcd}$$

The natural surjection $\pi : A \to A/I^2$ is one lift, so by Remark 2.10(3), all lifts have the form $\pi - f$ for $f \in \operatorname{Hom}_{A/I}(\Omega_{A/R} \otimes_A A/I, I/I^2)$. Such lifts $\pi - f$ are zero on I if and only if for all $a \in I$, $f(da)$ equals the image of a in I/I^2. That is, lifts in our original diagram correspond to $f \in \operatorname{Hom}_{A/I}(\Omega_{A/R} \otimes_A A/I, I/I^2)$ such that $f d^I_{A/R} = 1_{I/I^2}$. □

Theorem 2.17. *Suppose we have a map $R \to A$ and an ideal $I \subseteq A$. If $R \to A/I$ is quasi-smooth, then $d^I_{A/R}$ is a split injection. Conversely, if $d^I_{A/R}$ is a split injection and $R \to A$ is quasi-smooth, then $R \to A/I$ is quasi-smooth.*

Proof. Apply Lemma 2.5 and Lemma 2.16. □

Corollary 2.18. *Suppose $R \to A$ is quasi-smooth, $B = A/I$, and $S \subseteq B$ is a multiplicative set. Then $R \to B_S$ is quasi-smooth if and only if $d^I_{A/R} \otimes_B B_S$ is a split injection.*

Proof. Let $T \subseteq A$ be the preimage of S, so that $B_S = A_T/I_T$. By Remark 2.15(2), $d^I_{A/R} \otimes_B B_S = d^{I_T}_{A_T/R}$. Since $R \to A_T$ is quasi-smooth, this map is a split injection iff $R \to B_S$ is quasi-smooth, by Theorem 2.17. □

Corollary 2.19. *If $R \to B$ is quasi-smooth, then $\Omega_{B/R}$ is a projective B-module.*

Proof. Write $B = A/I$ for some polynomial ring $A = R[X_\lambda]$. (Use Remark 2.6.) By Example 2.11, $\Omega_{A/R}$ is a free A-module. So using Theorem 2.17, $d^I_{A/R} : I/I^2 \to \Omega_{A/R} \otimes_A B$ is a split injection into a free B-module. Therefore its cokernel, which is $\Omega_{B/R}$, is a projective B-module. □

An improvement of Corollary 2.19 can be found in [Sw, Theorem 3.4] where, after defining the module $\Gamma_{B/R}$, it is proved that $R \to B$ is quasi-smooth iff $\Omega_{B/R}$ is projective and $\Gamma_{B/R} = 0$. Merely having $\Omega_{B/R}$ projective certainly does not show $R \to B$ is quasi-smooth. Indeed, in the following example, take $I \subseteq R$ any ideal with $I \neq I^2$, since clearly $\Omega_{(R/I)/R} = 0$ is projective.

Example 2.20. $R \to R/I$ is quasi-smooth iff $I = I^2$. More generally, if $R \to A$ is quasi-smooth with $\Omega_{A/R} = 0$, and $I \subseteq A$ is an ideal, then $R \to A/I$ is quasi-smooth iff $I = I^2$. Indeed, by Theorem 2.17, $R \to A/I$ is quasi-smooth iff $d^I_{A/R}$ is a split injection; since $\Omega_{A/R} = 0$, this holds iff $I/I^2 = 0$.

3. Quasi-smooth Maps of Schemes

Definition 3.1. *We say a map of schemes $X \to S$ is* quasi-smooth *if for every ring C, ideal $J \subseteq C$ with $J^2 = 0$, and maps $\operatorname{Spec} C/J \to X$ and $\operatorname{Spec} C \to S$ making the diagram below commute, there exists a scheme map $\operatorname{Spec} C \to X$ making the diagram commute.*

$$\begin{tikzcd} X \arrow[d] & \operatorname{Spec} C/J \arrow[l] \arrow[d] \\ S & \operatorname{Spec} C \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

We postpone defining "smooth" for scheme maps until Section 8. For geometric interpretations of this definition, see Section 1. Clearly, a ring map $R \to B$ is quasi-smooth iff the associated scheme map $\operatorname{Spec} B \to \operatorname{Spec} R$ is quasi-smooth. It is also easy to see that the simple properties of Proposition 2.2(1),(2),(3) generalize to non-affine maps. Further, since $\operatorname{Spec} C/J$ and $\operatorname{Spec} C$ are equal as topological spaces, it follows directly from the definition that if $X \to S$ is quasi-smooth and $X' \subseteq X$ and $S' \subseteq S$ are open sets with the image of X' contained in S', then $X' \to S'$ is quasi-smooth. (In particular, taking $X = S = S'$, open immersions are quasi-smooth.) For a converse, see Theorem 4.11 and the discussion which ends Section 4.

To define a sheaf of differentials which generalizes our module of differentials, we will use the following lemma.

Lemma 3.2. *For a map of rings $R \to A$, let $T \subseteq R$ and $S \subseteq A$ be multiplicative sets such that elements of T map to units of A_S. Then $\Omega_{A/R} \otimes_A A_S = \Omega_{A_S/R_T}$. In particular, for $r \in R$ and $a \in A$, if the image of r in A_a is a unit, then $\Omega_{A/R} \otimes_A A_a = \Omega_{A_a/R_r}$.*

Proof. Using Example 2.13 and Example 2.12, we calculate $\Omega_{A_S/R_T} = \Omega_{A_T/R_T} \otimes_{A_T} A_S = (\Omega_{A/R} \otimes_A A_T) \otimes_{A_T} A_S$, which equals $\Omega_{A/R} \otimes_A A_S$. □

Definition/Proposition 3.3. *For a map of schemes $f : X \to S$, there exists a quasi-coherent sheaf $\Omega_{X/S}$ of $\mathcal{O}_X$-modules, called the* sheaf of differentials, *uniquely defined such that if we have open affine sets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$, then $\Omega_{X/S}|_U$ is the sheaf associated to the $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$. For $x \in X$, the stalk $(\Omega_{X/S})_x$ equals $\Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}$.*

Proof. To show this is well defined, we find natural isomorphisms which patch together the sheaves on U described above. For $i = 1, 2$, suppose $U_i \subseteq X$ and $V_i \subseteq S$ are open affine sets with $f(U_i) \subseteq V_i$. Let $\mathscr{F}_i$ be the sheaf of $\mathscr{O}_X|_{U_i}$-modules associated to the $\mathscr{O}_X(U_i)$-module $\Omega_{\mathscr{O}_X(U_i)/\mathscr{O}_S(V_i)}$. We will show that there is a natural isomorphism $\mathscr{F}_1|_{U_1\cap U_2} \cong \mathscr{F}_2|_{U_1\cap U_2}$. Consider all open affine $U_\lambda \subseteq X$ and $V_\lambda \subseteq S$ such that $f(U_\lambda) \subseteq V_\lambda$, U_λ is a distinguished open set of both U_1 and U_2, and V_λ is a distinguished open set of both V_1 and V_2. The set of all such U_λ form a basis of $U_1 \cap U_2$. By the last statement of Lemma 3.2, on such U_λ, the sections of both $\mathscr{F}_1$ and $\mathscr{F}_2$ are $\Omega_{\mathscr{O}_X(U_\lambda)/\mathscr{O}_S(V_\lambda)}$. This gives us the desired natural isomorphism. The computation of stalks follows from the main statement of Lemma 3.2. □

Proposition 3.4. *Suppose we have a commutative diagram of schemes as follows, with $Y' \subseteq Y$ a closed subscheme defined by a sheaf of ideals $\mathscr{J}$ with $\mathscr{J}^2 = 0$.*

$$\begin{tikzcd} X \arrow[d] & Y' \arrow[l] \arrow[d] \\ S & Y \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

If $g : Y \to X$ is a lift as indicated, then there is a bijective correspondence between the set of all such lifts and $\mathrm{Hom}_{Y'}(\Omega_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'}, \mathscr{J})$. If g' is another lift, this correspondence is given by "$g - g'$," by which we mean that over open affine sets of Y and X whose images are contained in a single open affine set of S, we subtract the ring maps corresponding to g and g' and use the correspondence of Remark 2.10(3).

Proof. Use patching and Remark 2.10(3), which is the affine case of this result. □

Remark 3.5. In the following corollary, the condition $\Omega_{B/R} = 0$ [resp. $\Omega_{X/S} = 0$] will be shown equivalent to uniqueness of lifts, as opposed to the existence of lifts which defines quasi-smoothness. Terminology varies, but this condition can be called unramified, 0-unramified, or formally unramified. When lifts exist and are unique, i.e. we have quasi-smoothness and $\Omega_{B/R} = 0$ [resp. $\Omega_{X/S} = 0$], this can be called étale, 0-étale, formally étale, or quasi-étale. We will avoid these terms, but see the definitions in [Mat1, 38.E], [Mat2, p. 193], [AK, (VI, 3.1 and 3.3)], [EGA, 0_{IV}.19.10.2], and [Sw, p. 136].

Corollary 3.6.

(1) *For a map of rings $R \to B$, $\Omega_{B/R} = 0$ if and only if for every ring C and ideal $J \subseteq C$ with $J^2 = 0$, there exists* at most one *map as indicated below in the first diagram.*

(2) *For a map of schemes $X \to S$, $\Omega_{X/S} = 0$ if and only if for every ring C and ideal $J \subseteq C$ with $J^2 = 0$, there exists* at most one *map as indicated below in the second diagram.*

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dashed] & C/J \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d] & \mathrm{Spec}\, C/J \arrow[l] \arrow[d] \\ S & \mathrm{Spec}\, C \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

Proof. Clearly (1) follows from (2). To show (2), first suppose $\Omega_{X/S} = 0$ and we have the outer maps of the second diagram above, with $J^2 = 0$. We denote by $\tilde{J}$ the sheaf of $\mathscr{O}_{\mathrm{Spec}\, C/J}$-modules defined by J. If one lift exists, then by Proposition 3.4, the set of all lifts are in bijective correspondence with $\mathrm{Hom}_{\mathrm{Spec}\, C/J}(\Omega_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{\mathrm{Spec}\, C/J}, \tilde{J}) = 0$, so there can only be one lift.

Conversely, suppose $\Omega_{X/S} \neq 0$. Find open affine sets $\operatorname{Spec} B \subseteq X$ and $\operatorname{Spec} R \subseteq S$ such that the image of $\operatorname{Spec} B$ is contained in $\operatorname{Spec} R$ and $\Omega_{B/R} \neq 0$. Let $C = (\operatorname{Sym}^{\bullet}_B \Omega_{B/R})/I$, where I is generated by all homogeneous elements of degree at least 2. If $J \subseteq C$ is the set of homogeneous elements of degree 1, then J is an ideal, $J^2 = 0$, $C/J = B$, and $J \cong \Omega_{B/R}$ as B-modules. Then inclusion of degree zero gives us the existence of one lift $B \to C$ in the first diagram above. This induces $\operatorname{Spec} C \to \operatorname{Spec} B \hookrightarrow X$, which is a lift in the second diagram above. By Proposition 3.4, the set of all lifts are in bijective correspondence with $\operatorname{Hom}_{\operatorname{Spec} C/J}(\Omega_{X/S} \otimes_{\mathcal{O}_X} \operatorname{Spec} C/J, \tilde{J})$. Using $C/J = B$ and $J \cong \Omega_{B/R}$, this is isomorphic to the B-module endomorphisms of $\Omega_{B/R}$. Since $\Omega_{B/R} \neq 0$, the identity and zero maps are distinct endomorphisms, so we conclude that the second diagram above must have more than one lift. This completes the corollary. $\square$

4. The Local Nature of Quasi-smoothness

In this section we will look at the local nature of quasi-smoothness. Local can mean at points—prime ideals—for which we will have Theorem 4.7 (affine) and Corollary 4.13 (non-affine), or local can mean on open sets, for which we prove Theorem 4.11 (non-affine) and Corollary 4.12 (affine). Also see Proposition 5.4.

If B is an R-algebra, we can get many results if $\Omega_{B/R}$ is a finitely presented B-module by using the following well known result, which can be found in [Mat2, Theorem 7.11] or [L, Proposition I.2.13].

Proposition 4.1. *For $B \to B_0$ a flat map, let M and N be B-modules with M finitely presented. Then the natural map $\operatorname{Hom}_B(M, N) \otimes_B B_0 \to \operatorname{Hom}_{B_0}(M \otimes_B B_0, N \otimes_B B_0)$ is an isomorphism.*

The condition of being finitely presented, however, is rather restrictive, so we introduce the following generalization.

Definition 4.2. *We say a B-module M is* sum finitely presented *if there exists a B-module M' such that $M \oplus M'$ is a direct sum (possibly infinite) of finitely presented B-modules.*

Any free module is certainly the sum of free modules of rank 1, from which we conclude that a projective module is sum finitely presented. More generally, see Lemma 5.3(1).

Lemma 4.3. *Let M, Y, and Z be B-modules, with M sum finitely presented. Suppose we have B-module maps $f : M \to Z$ and $g : Y \to Z$. Then either of the following conditions implies f is in the image of $\operatorname{Hom}_B(M, Y) \xrightarrow{g} \operatorname{Hom}_B(M, Z)$, i.e., the following map exists.*

$$\begin{tikzcd} & M \arrow[dl, dashed] \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

(1) For all maximal ideals $\mathfrak{M} \subseteq B$, we have that $f \otimes_B B_{\mathfrak{M}}$ is in the image of the induced map $\operatorname{Hom}_{B_{\mathfrak{M}}}(M_{\mathfrak{M}}, Y_{\mathfrak{M}}) \xrightarrow{g} \operatorname{Hom}_{B_{\mathfrak{M}}}(M_{\mathfrak{M}}, Z_{\mathfrak{M}})$.

(2) For some faithfully flat B-algebra B', we have that $f \otimes_B B'$ is in the image of the induced map $\operatorname{Hom}_{B'}(M \otimes_B B', Y \otimes_B B') \xrightarrow{g} \operatorname{Hom}_{B'}(M \otimes_B B', Z \otimes_B B')$.

Proof. For fixed $g : Y \to Z$, the hypotheses and conclusion hold for every $f_\alpha : M_\alpha \to Z$ iff they hold for $\oplus f_\alpha : \oplus M_\alpha \to Z$. Also, the hypotheses and conclusion trivially hold if f is the zero map. Therefore, replacing $f : M \to Z$ by $f \oplus 0 : M \oplus M' \to Z$, we may assume M is the direct sum of finitely presented modules. But verifying the conclusion separately for each direct summand, without loss of generality, M is finitely presented.

Let Q be the cokernel of $\mathrm{Hom}_B(M,Y) \xrightarrow{g} \mathrm{Hom}_B(M,Z)$, and let $\phi : B \to Q$ be the B-module map sending 1 to the image of f. By Proposition 4.1, for any flat map $B \to B_0$, $Q \otimes_B B_0$ is the cokernel of $\mathrm{Hom}_{B_0}(M \otimes_B B_0, Y \otimes_B B_0) \xrightarrow{g} \mathrm{Hom}_{B_0}(M \otimes_B B_0, Z \otimes_B B_0)$. Therefore. in (2) [resp. (1)], we have $\phi \otimes_B B' = 0$ [resp. $\phi \otimes_B B_{\mathfrak{M}} = 0$ for all maximal ideals $\mathfrak{M} \subseteq B$]. Thus $\phi = 0$, which is what we need. □

Lemma 4.4. *Let d be a B-module map such that its cokernel is a sum finitely presented B-module. Then d is a split injection if either of the following conditions holds.*

(1) For all maximal ideals $\mathfrak{M} \subseteq B$, $d \otimes_B B_{\mathfrak{M}}$ is a split injection.

(2) For some faithfully flat B-algebra B', $d \otimes_B B'$ is a split injection.

Proof. Say $d : X \to Y$, and Z is the cokernel of d, with $g : Y \to Z$ the natural map. Our result follows from Lemma 4.3, using $M = Z$ and $f = 1_Z$, once we observe that d [resp. $d \otimes_B B_{\mathfrak{M}}$, $d \otimes_B B'$] is a split injection if and only if the conclusion [resp. (1), (2)] of Lemma 4.3 holds. □

We will use the following lemma several times. In our applications, $\mathrm{Tor}_1^R(A/I, R') = 0$ will be satisfied in some cases when $R \to A/I$ is flat and in other cases when $R \to R'$ is flat.

Lemma 4.5. *Let A and R' be R-algebras, $I \subseteq A$ an ideal, $A' = A \otimes_R R'$, and $I' = IA'$. If $\mathrm{Tor}_1^R(A/I, R') = 0$, then the canonical surjections $I \otimes_R R' \twoheadrightarrow I'$ and $I/I^2 \otimes_R R' \twoheadrightarrow I'/(I')^2$ are isomorphisms.*

Proof. The sequence $0 \to I \to A \to A/I \to 0$ is exact, and tensoring with R', so is the following: $0 \to I \otimes_R R' \to A' \to A'/I' \to 0$, where the first zero is $\mathrm{Tor}_1^R(A/I, R')$. Thus $I \otimes_R R' \cong I'$. Furthermore, $I/I^2 \otimes_R R'$ is the cokernel of $I^2 \otimes_R R' \to I \otimes_R R'$, but the image of this map in $I \otimes_R R' \cong I'$ is clearly $(I')^2$. □

Proposition 4.6. *Let $R \to B$ be a map such that $\Omega_{B/R}$ is a sum finitely presented B-module. Then either of the following conditions is sufficient to show that $R \to B$ is quasi-smooth.*

(1) $R \to B_{\mathfrak{M}}$ is quasi-smooth for all maximal ideals $\mathfrak{M} \subseteq B$.

(2) $R' \to B \otimes_R R'$ is quasi-smooth for some faithfully flat R-algebra R'.

Proof. Write $B = R[X_\lambda]/I$ as in Remark 2.6. By Theorem 2.17, it suffices to show that $d = d^I_{R[X_\lambda]/R}$ is a split injection; note that the cokernel of d is $\Omega_{B/R}$. For (1), by Corollary 2.18, $d \otimes_B B_{\mathfrak{M}}$ is a split injection for all maximal ideals $\mathfrak{M} \subseteq B$. Thus d is a split injection by Lemma 4.4. For (2), let $B' = B \otimes_R R'$ and $I' = IR'[X_\lambda]$, so that $B' = R'[X_\lambda]/I'$. By Remark 2.15(1), $d \otimes_B B'$ equals the canonical surjection $I/I^2 \otimes_B B' \twoheadrightarrow I'/(I')^2$ followed by $d^{I'}_{B'/R'}$. The latter map is a split injection by Theorem 2.17, and the former is an isomorphism by Lemma 4.5. Therefore $d \otimes_B B'$ is a split injection, and we apply Lemma 4.4 to get that d is a split injection. □

The following result shows how affine quasi-smoothness is local, in the sense of looking at points, that is, local rings. This extends [EGA, 0_{IV}.22.6.6], which only proves equivalence for $R \to B$ conormally finite. (See our definition in Section 5.)

Theorem 4.7. *For a map of rings $R \to B$, consider the following conditions.*

(a) $R \to B$ is quasi-smooth.

(b) For all maximal ideals $\mathfrak{m} \subseteq R$, $R_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is quasi-smooth.

(c) For all maximal ideals $\mathfrak{M} \subseteq B$, $R \to B_{\mathfrak{M}}$ is quasi-smooth.

(d) For all maximal ideals $\mathfrak{M} \subseteq B$ with $\mathfrak{p} = \mathfrak{M} \cap R$, $R_{\mathfrak{p}} \to B_{\mathfrak{M}}$ is quasi-smooth.

Then we have (a)⇒(b)⇒(c)⇔(d), and if $\Omega_{B/R}$ is sum finitely presented, then all of the conditions are equivalent.

Proof. (a)⇒(b) is a change of base; use Proposition 2.4 for (c)⇔(d). For (b)⇒(d), let $\mathfrak{M} \subseteq B$ be maximal with $\mathfrak{p} = \mathfrak{M} \cap R$, and let $\mathfrak{m} \subseteq R$ be a maximal ideal containing $\mathfrak{p}$. By assumption, $R_\mathfrak{m} \to B_\mathfrak{m}$ is quasi-smooth. Composing with the localization $B_\mathfrak{m} \to B_\mathfrak{M}$, we get that $R_\mathfrak{m} \to B_\mathfrak{M}$ is quasi-smooth, and then we apply Proposition 2.4, to conclude that $R_\mathfrak{p} \to B_\mathfrak{M}$ is quasi-smooth. Finally, if $\Omega_{B/R}$ is sum finitely presented, then Proposition 4.6(1) gives (c)⇒(a). □

In the following example we define ring maps $R \to R^\Lambda \to B$ with the following properties. First, $R^\Lambda_\mathfrak{p} \to B_\mathfrak{p}$ is quasi-smooth for all prime ideals $\mathfrak{p} \subseteq R^\Lambda$, but $R^\Lambda \to B$ is not quasi-smooth. This shows that (b) does not imply (a) in Theorem 4.7. Also we show that (c) does not imply (b), because $R \to B_\mathfrak{P}$ is quasi-smooth for all prime ideals $\mathfrak{P} \subseteq B$, but for all prime ideals $\mathfrak{p}_0 \subseteq R$, $R_{\mathfrak{p}_0} \to B_{\mathfrak{p}_0}$ is not quasi-smooth. In these examples, the key point is to have a module of differentials which is not projective, but yet is projective at every prime ideal of B. We could use the ring of [AB, Exercise 4, pp. 337–8], but the important point of that ring is that it has infinitely many idempotents, an idea we exploit in the following example.

Example 4.8. For any non-zero ring R, let $R^\Lambda = R[X_\lambda]/(X_\lambda^2 - X_\lambda)$, where the λ are in a fixed infinite index set Λ. By Remark 2.10(2), $\Omega_{R^\Lambda/R} = 0$. Let $\mathfrak{p} \subseteq R^\Lambda$ be a prime ideal. Then for all λ, in $R^\Lambda_\mathfrak{p}$, X_λ equals 0 or 1, since local rings have no nontrivial idempotents. (Indeed, if a is idempotent, consider a and $1 - a$. Since their product is zero, one of them is in the maximal ideal, which makes the other a unit, which in turn makes the original one zero. So $a = 0$ or 1.) So as an R-algebra, $R^\Lambda_\mathfrak{p}$ is a localization of R, and $R \to R^\Lambda_\mathfrak{p}$ is thus quasi-smooth. Therefore $R \to R^\Lambda$ is quasi-smooth by Theorem 4.7.

Let $\mathfrak{a} \subseteq R^\Lambda$ be the ideal generated by all the X_λ. For a prime ideal $\mathfrak{p} \subseteq R^\Lambda$, $\mathfrak{a}_\mathfrak{p} = 0$ iff $\mathfrak{p} \supseteq \mathfrak{a}$, and otherwise $\mathfrak{a}_\mathfrak{p} = R^\Lambda_\mathfrak{p}$. Taking a single new variable Y, let $B = R^\Lambda[Y]/\mathfrak{a}Y$, so that $B_\mathfrak{p} = R^\Lambda_\mathfrak{p}[Y]$ iff $\mathfrak{p} \supseteq \mathfrak{a}$, and otherwise $B_\mathfrak{p} = R^\Lambda_\mathfrak{p}$. In either case, $R^\Lambda_\mathfrak{p} \to B_\mathfrak{p}$ is flat and quasi-smooth. (Thus $R^\Lambda \to B$ is flat, which we need in the paragraph following Theorem 5.11.) So for $\mathfrak{M} \subseteq B$ maximal, $R^\Lambda \to B_\mathfrak{M}$ is quasi-smooth by (b)⇒(c) of Theorem 4.7. Using localization and composition, we get that for all $\mathfrak{P} \subseteq B$ prime, $R^\Lambda \to B_\mathfrak{P}$ and $R \to B_\mathfrak{P}$ are quasi-smooth.

Being the cokernel of $d^{\mathfrak{a}Y}_{R^\Lambda[Y]/R^\Lambda}$, $\Omega_{B/R^\Lambda} \cong B/\mathfrak{a}B$. If $B/\mathfrak{a}B$ were a projective module, then $B \cong \mathfrak{a}B \oplus B/\mathfrak{a}B$, and $\mathfrak{a}B$ would be principal. But in the map $B \to B/YB = R^\Lambda$, we have $\mathfrak{a}B \twoheadrightarrow \mathfrak{a}$. Since $\mathfrak{a}$ is not even finitely generated, we conclude that Ω_{B/R^Λ} is not a projective module. Also, by Remark 2.10(2), $\Omega_{B/R^\Lambda} = \Omega_{B/R}$, so neither $R \to B$ nor $R^\Lambda \to B$ is quasi-smooth by Corollary 2.19. If $\mathfrak{p}_0 \subseteq R$ is a prime ideal, then repeating this construction for $R = R_{\mathfrak{p}_0}$ gives that $R_{\mathfrak{p}_0} \to B_{\mathfrak{p}_0}$ is not quasi-smooth. This completes what we wanted to prove in this example. For later use in Example 5.6, let $\mathfrak{p} \subseteq R^\Lambda$ be a prime ideal containing $\mathfrak{a}$. Note that although $\mathfrak{a}_\mathfrak{p} = 0$, nonetheless $\mathfrak{a}_s \neq 0$ for all $s \in R^\Lambda - \mathfrak{p}$, because choosing some X_{λ_0} which does not appear in s, we have $X_{\lambda_0} \neq 0$ in R^Λ_s.

Now let us look at how quasi-smoothness is local, in the sense of open sets. In this case, we will begin by looking at non-affine schemes. Similar to Theorem 4.7 where we required $\Omega_{B/R}$ to be sum finitely presented, we will use in the non-affine case a similar condition on $\Omega_{X/S}$, for which we make the following definitions.

Definition 4.9. *Let X be a scheme and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$-modules. As in* [EGA, 0_I.5.2], *we say $\mathscr{F}$ is* finite type *[resp.* finitely presented*] if for all sets U in some open cover of X, $\mathscr{F}|_U$ is* *the cokernel of some map $\mathscr{G} \to \mathcal{O}_X^n|_U$, for $\mathscr{G}$ a sheaf on U and n a positive integer [resp. the cokernel of some map $\mathcal{O}_X^m|_U \to \mathcal{O}_X^n|_U$, for m and n positive integers]. If $\mathscr{F}$ is quasi-coherent, this is equivalent to $\mathscr{F}(U)$ being a finitely generated [resp. finitely presented] $\mathcal{O}_X(U)$-module for all affine U, and it suffices to check this on an open affine cover of X. Let us define $\mathscr{F}$ to be* sum finitely presented *if there exists a sheaf $\mathscr{F}'$ such that $\mathscr{F} \oplus \mathscr{F}'$ is the direct sum (possibly infinite) of finitely presented sheaves.*

Note that every finitely presented sheaf is *a priori* quasi-coherent, and that direct sums and direct summands of quasi-coherent sheaves are quasi-coherent [EGA, 0_{I}.5.1.3 and I.1.4.1]. In particular, a sum finitely presented sheaf is quasi-coherent.

Recall that on an affine scheme $Z = \operatorname{Spec} A$, with $\mathfrak{U} = \{\operatorname{Spec} A_{a_i}\}$ a finite open cover by distinguished open sets, all Čech cohomology groups $\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H})$ are zero for $p \geq 1$, if $\mathscr{H}$ is a quasi-coherent sheaf of $\mathcal{O}_Z$-modules. (See [H] to define Čech cohomology and [EGA, III.1.2.4] for this result.) We will weaken the hypothesis that $\mathscr{H}$ is quasi-coherent, because we need this result for $\mathscr{H} = \mathscr{H}om_{\mathcal{O}_Z}(\mathscr{F}, \mathscr{G})$. Note that $\mathscr{H}om_{\mathcal{O}_Z}(\mathscr{F}, \mathscr{G})$ is quasi-coherent if $\mathscr{F}$ is finitely presented and $\mathscr{G}$ is quasi-coherent [EGA, I.9.1.1]. (It is not hard to see that $\mathscr{H}om_{\mathcal{O}_Z}(\mathscr{F}, \mathscr{G})$ fails to be quasi-coherent even for $\mathscr{F}$ free of infinite rank.) We will need the following result for $p = 1$.

Lemma 4.10. *With Z and $\mathfrak{U}$ as above, suppose $\mathscr{F}$ and $\mathscr{G}$ are sheaves of quasi-coherent $\mathcal{O}_Z$-modules, with $\mathscr{F}$ sum finitely presented. Then for $p \geq 1$, $\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}om_{\mathcal{O}_Z}(\mathscr{F}, \mathscr{G})) = 0$.*

Proof. For notation, let $\mathscr{H}(\cdot) = \mathscr{H}om_{\mathcal{O}_Z}(\cdot, \mathscr{G})$. If $\mathscr{F} \oplus \mathscr{F}' = \oplus_\alpha \mathscr{F}_\alpha$ with all $\mathscr{F}_\alpha$ finitely presented, then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}(\mathscr{F})) \oplus \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}(\mathscr{F}')) = \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}(\mathscr{F} \oplus \mathscr{F}')) = \check{\mathrm{H}}^p(\mathfrak{U}, \Pi_\alpha \mathscr{H}(\mathscr{F}_\alpha)) = \Pi_\alpha \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}(\mathscr{F}_\alpha)) = 0$, since each $\mathscr{H}(\mathscr{F}_\alpha)$ is quasi-coherent. □

Theorem 4.11. *Let $f : X \to S$ be a map of schemes such that we have one of the following conditions.*

(1) $\Omega_{X/S}$ is a sum finitely presented sheaf.
(2) $\Omega_{X/S}$ is the direct sum of sheaves of finite type.

Suppose we have pairs of open sets $X_\alpha \subseteq X$ and $S_\alpha \subseteq S$ such that for every α, $f(X_\alpha) \subseteq S_\alpha$ and the restriction $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is quasi-smooth. If the X_α cover X, then f is quasi-smooth.

Proof. First we show (1). Suppose we have a commutative diagram as follows, where Y is affine and $Y' \subseteq Y$ is a closed subscheme defined by a sheaf of ideals $\mathscr{J}$ with $\mathscr{J}^2 = 0$. To show f quasi-smooth, it suffices to find a lift as indicated.

$$\begin{tikzcd} X \arrow[d] & Y' \arrow[l] \arrow[d] \\ S & Y \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

Find an open cover $\{Y_i\}$ of Y and indices α_i such that letting $Y_i' = Y' \cap Y_i$, the image of Y_i [resp. Y_i'] is contained in S_{α_i} [resp. X_{α_i}]. By refining the cover, we may assume the cover is a finite affine cover by distinguished open sets. Let $\mathfrak{U} = \{Y_i'\}$ be the induced cover on Y'. In the first diagram below, find lifts as indicated and define $g_i : Y_i \to X$ to be the composition of these lifts with the inclusion $X_{\alpha_i} \hookrightarrow X$.

$$\begin{tikzcd} X \arrow[d] & X_{\alpha_i} \arrow[l, hook'] \arrow[d] & Y_i' \arrow[l] \arrow[d] \\ S & S_{\alpha_i} \arrow[l, hook'] & Y_i \arrow[l] \arrow[ul, dashed] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d] & Y_i' \cap Y_j' \arrow[l] \arrow[d] \\ S & Y_i \cap Y_j \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

Let $\mathscr{H} = \mathscr{H}om_{Y'}(\Omega_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'}, \mathscr{J})$. Certainly $\Omega_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'}$ is sum finitely presented since $\Omega_{X/S}$ is, and thus by Lemma 4.10, $\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{H}) = 0$. For indices i and j, g_i and g_j both define lifts of the second diagram above, so by Proposition 3.4, $g_i - g_j$ defines a section of the sheaf $\mathscr{H}$ on the open set $Y_i' \cap Y_j'$. Such $g_i - g_j$ give a 1-cocycle because on $Y_i' \cap Y_j' \cap Y_k'$, $(g_j - g_k) - (g_i - g_k) + (g_i - g_j) = 0$. Since $\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{H}) = 0$, the $g_i - g_j$ are a 1-coboundary, say h_i are sections of $\mathscr{H}$ on Y_i' with $h_i - h_j = g_i - g_j$ on $Y_i' \cap Y_j'$. Then by Proposition 3.4 again, $g_i - h_i$ give lifts $Y_i \to X$. Restricting to maps $Y_i \cap Y_j \to X$, $g_i - h_i = g_j - h_j$, and so we can patch together the $g_i - h_i$ to get a lift $Y \to X$ as desired, proving $X \to S$ is quasi-smooth.

Now suppose (2) holds, say $\Omega_{X/S} = \oplus \mathscr{F}_\lambda$, with each $\mathscr{F}_\lambda$ finite type. Each $\mathscr{F}_\lambda$ is quasi-coherent, because it is a direct summand of the quasi-coherent sheaf $\Omega_{X/S}$. Fix open affine sets $U \subseteq X_\alpha$ and $V \subseteq S_\alpha$, such that $f(U) \subseteq V$. (Allowing U, V, and α to vary, such U cover X.) By Corollary 2.19, $\Omega_{X/S}(U)$ is a projective $\mathscr{O}_X(U)$-module, and so each $\mathscr{F}_\lambda(U)$ is a finitely generated projective module, thus finitely presented. So each $\mathscr{F}_\lambda$ is a finitely presented sheaf, and this tells us (1) holds. □

Corollary 4.12. *Let $R \to B$ be a map of rings such that we have one of the following conditions.*

(1) $\Omega_{B/R}$ is a sum finitely presented B-module.

(2) $\Omega_{B/R}$ is the direct sum of finitely generated B-modules.

Suppose $b_1, \ldots, b_n$ are elements of B which generate the unit ideal. If all $R \to B_{b_i}$ are quasi-smooth, then $R \to B$ is quasi-smooth.

This corollary follows directly from Theorem 4.11, although (1) is also a consequence of Theorem 4.7. Note that in Theorem 4.7, we can not substitute the hypothesis that $\Omega_{B/R}$ is the direct sum of finitely generated B-modules, because in Example 4.8, our modules of differentials were generated by a single element. Thus we see one difference between looking at the local nature of quasi-smoothness in terms of points (prime ideals) and of open sets. Having looked at quasi-smoothness for non-affine schemes, we now prove non-affine versions of Theorem 4.7 and Proposition 4.6(2).

Corollary 4.13. *For a map of schemes $X \to S$ consider the following conditions.*

(a) $X \to S$ is quasi-smooth.

(b) For all closed points $s \in S$, $X \times_S \operatorname{Spec} \mathscr{O}_{S,s} \to \operatorname{Spec} \mathscr{O}_{S,s}$ is quasi-smooth.

(c) For all closed points $x \in X$, $\operatorname{Spec} \mathscr{O}_{X,x} \to S$ is quasi-smooth.

(d) For all closed points $x \in X$, if $s \in S$ is its image, then $\operatorname{Spec} \mathscr{O}_{X,x} \to \operatorname{Spec} \mathscr{O}_{S,s}$ is quasi-smooth.

Then we have (a)⇒(b)⇒(c)⇔(d), and if $\Omega_{X/S}$ is sum finitely presented, then all of the conditions are equivalent.

Proof. (a)⇒(b) is a change of base. To show (b)⇒(c)⇔(d), find an affine neighborhood of x which maps into an affine neighborhood of its image $s \in S$, and then apply Theorem 4.7. If $\Omega_{X/S}$ is sum finitely presented, then we get (d)⇒(a) by using Theorem 4.11 to reduce to a map of affine schemes; then apply Theorem 4.7. □

Corollary 4.14. *Let X and S' be schemes over S with $\Omega_{X/S}$ sum finitely presented and $S' \to S$ flat and surjective. Then $X \times_S S' \to S'$ is quasi-smooth if and only if $X \to S$ is quasi-smooth.*

Proof. Since "if" is simply a base change, suppose $X \times_S S' \to S'$ is quasi-smooth. Let an arbitrary $x \in X$ have image $s \in S$. Choose $s' \in S'$ which has image s, so that $\mathcal{O}_{S,s} \to \mathcal{O}_{S',s'}$ is faithfully flat. Now, through base change and localization, $\operatorname{Spec} \mathcal{O}_{X,x} \times_S \operatorname{Spec} \mathcal{O}_{S',s'} \to \operatorname{Spec} \mathcal{O}_{S',s'}$ is quasi-smooth, whence $\mathcal{O}_{S',s'} \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ is quasi-smooth. By Proposition 4.6(2), $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is quasi-smooth, so by Corollary 4.13, $X \to S$ is quasi-smooth. □

We now look at which hypotheses are necessary in the results of this section. Our sharpest result is Theorem 4.7, the affine case of quasi-smoothness at points, where we required $\Omega_{B/R}$ to be sum finitely presented in order to show $R \to B$ quasi-smooth. Not only did we see in Example 4.8 that the conclusions can fail in the absence of this hypothesis, but this hypothesis is necessary by Corollary 2.19: if $R \to B$ is quasi-smooth, then $\Omega_{B/R}$ is projective, and thus sum finitely presented. On the other hand, in Corollary 4.13, the non-affine version of this result, it is open whether $\Omega_{X/S}$ must be sum finitely presented. That is, for $X \to S$ quasi-smooth, it is unclear whether $\Omega_{X/S}$ is *globally* a direct summand of a sum of finitely presented sheaves.

When we looked at quasi-smoothness over open covers, in our affine result, Corollary 4.12, condition (1) is necessary by Corollary 2.19, just as above; but it is open whether (2) is also necessary. If it were not, $\Omega_{B/R}$ would be a non-finitely generated projective module which is not the direct sum of finitely generated modules. This can not happen if B is Noetherian [B, Corollary 4.4].

Finally, we consider non-affine quasi-smoothness, where local means over open sets. In [EGA, IV.17.1.6], Grothendieck gives the following proposition. (For consistency we use the term quasi-smooth rather than Grothendieck's equivalent formally smooth.) Let $f : X \to S$ be a map of schemes. Suppose we have an open cover X_α of X [resp. S_β of S] such that all $X_\alpha \to S$ [resp. all $f^{-1}(S_\beta) \to S_\beta$] are quasi-smooth. Then $X \to S$ is quasi-smooth. There is an error, however, because by closely examining Grothendieck's proof, we see he uses one additional hypothesis, namely that the sheaf $\Omega_{X/S}$ is finitely presented. Indeed, this hypothesis is needed when he applies Proposition IV.16.5.17. (Also see Corollary IV.16.5.18.) We generalized this proof to get Theorem 4.11, which weakens the hypothesis on $\Omega_{X/S}$ in two ways. However, let us ask whether hypotheses (1) and (2) are superfluous in Theorem 4.11, which is equivalent to whether [EGA, IV.17.1.6] is true as stated.

Open Question 4.15. *Suppose $X \to S$ is a map of schemes, X has open cover $\{X_\alpha\}$, and the image of each X_α lies in some open $S_\alpha \subseteq S$. If each $X_\alpha \to S_\alpha$ is quasi-smooth, is $X \to S$ quasi-smooth?*

We now consider Open Question 4.15 for X and S affine. If $\operatorname{Spec} B \to \operatorname{Spec} R$ existed showing the negative answer for Open Question 4.15, then $\Omega_{B/R}$ would be projective on an open cover, but not projective. To see non-projectivity, write $B = R[X_\lambda]/I$ as in Remark 2.6. Then $d^I_{R[X_\lambda]/R}$ is injective, since it is locally a split injection by Corollary 2.18. Since it is not itself a split injection, its cokernel $\Omega_{B/R}$ must not be projective. Conversely, a module which is projective on an open cover, but not globally shows the negative answer for Open Question 4.15, by Example 2.7. Thus Open Question 4.15 restricted to affine schemes is equivalent to the following question.

Open Question 4.16. *Suppose we have a ring R, r_i in R generating the unit ideal, and M an R-module such that M_{r_i} is a projective R_{r_i}-module for all i. Is M projective?*

It is well known that the answer to Open Question 4.16 is yes for M finitely presented; this can be generalized to M sum finitely presented by Lemma 4.3. In fact, Lemma 4.3 tells us that for M sum finitely presented, if $M_{\mathfrak{M}}$ is projective for all maximal ideals $\mathfrak{M}$, then M is projective, though this would be false for arbitrary M, as we see by looking at $\Omega_{B/R}$ in Example 4.8.

To look instead at Open Question 4.15 from a non-affine point of view, we could ask the following question, which would have an affirmative answer if Open Question 4.15 did.

Open Question 4.17. *Suppose $X \to S$ is a map of schemes such that for all open affine $U \subseteq X$ (or for all affine $U \subseteq X$ whose image is contained in an open affine set of S), the restriction $U \to S$ is quasi-smooth. Is $X \to S$ quasi-smooth?*

We also ask if the condition of sum finitely presented can be dropped in Corollary 4.14.

Open Question 4.18. *Suppose X and S' are schemes over S with $S' \to S$ flat and surjective. If $X \times_S S' \to S'$ is quasi-smooth, then is $X \to S$ quasi-smooth?*

Finally, just as Open Question 4.16 is equivalent to the affine case of Open Question 4.15, the following is equivalent to the affine case of Open Question 4.18. (The parallel argument to show equivalence requires Remark 2.15(1) and Lemma 4.5 in order to show $d^I_{R[X_\lambda]/R}$ injective.) Also, similar to Open Question 4.16, the following has an affirmative answer for M sum finitely presented, by Lemma 4.3.

Open Question 4.19. *Suppose M is an R-module and $R \to R'$ is faithfully flat. If $M \otimes_R R'$ is a projective R'-module, is M a projective R-module?*

Note that Open Question 4.16 is the special case of Open Question 4.19 with $R' = \oplus R_{r_i}$.

5. When $R \to B$ Is Conormally Finite

Definition 5.1. *We say a map of rings $R \to B$ is* conormally finite *if as an R-algebra, $B = A/I$ for some quasi-smooth R-algebra A and $I \subseteq A$ an ideal such that I/I^2 is a finitely generated B-module.*

Remark 5.2. For $R \to B$ conormally finite, $\Omega_{B/R}$ is sum finitely presented. Indeed, let A and I be as in the definition above. By Corollary 2.19, $\Omega_{A/R}$ is A-projective, so the differential map $d^I_{A/R} : I/I^2 \to \Omega_{A/R} \otimes_A B$ is a B-module map from a finitely generated module into a projective module. This tells us $\Omega_{B/R}$ is sum finitely presented, by Lemma 5.3(1). In Example 5.6 we will see a map $R \to B$ which is not conormally finite, but for which $\Omega_{B/R}$ is sum finitely presented.

Lemma 5.3. *Let $f : M \to P$ be a B-module map, with M finitely generated and P projective.*

1. *The cokernel of f is sum finitely presented.*
2. *If $S \subseteq B$ is a multiplicative set such that $f \otimes_B B_S$ is a split injection, then there exists $s \in S$ such that $f \otimes_B B_s$ is a split injection.*

Proof. Say $Q \oplus P$ is a free module. Then $Q \oplus \operatorname{coker} f$ is a quotient of a free module by a finitely generated submodule. Such a submodule lies in a direct summand which is generated by finitely many basis elements of the free module. So we see $Q \oplus \operatorname{coker} f$ is the direct sum of a single finitely presented module and free modules of rank 1. This shows (1).

For (2), It is easy to see that for any module N'', a map $g : N \to N'$ is a split injection if and only if $g \oplus 0 : N \to N' \oplus N''$ is a split injection. Applying this one time on both hypothesis and conclusion, we replace P by $Q \oplus P$, and thus assume P is free. Then the image of M is contained in a direct summand of P which is free of finite rank, so applying this once again, we assume P itself is free of finite rank. Now find $s_1 \in S$ and $h : P_{s_1} \to M_{s_1}$ such that

$(h \otimes_{B_{s_1}} B_S)(f \otimes_B B_S)$ is the identity map on M_S. Since M is finitely generated, find $s_2 \in S$ such that $(h \otimes_{B_{s_1}} B_{s_1 s_2})(f \otimes_B B_{s_1 s_2})$ is the identity on $M_{s_1 s_2}$, which shows $f \otimes_B B_{s_1 s_2}$ is a split injection. □

Proposition 5.4. *Suppose $R \to B$ is conormally finite. If $S \subseteq B$ is a multiplicative set such that $R \to B_S$ is quasi-smooth, then there exists $s \in S$ such that $R \to B_s$ is quasi-smooth.*

Proof. Write $B = A/I$, with A a quasi-smooth R-algebra and $I \subseteq A$ an ideal with I/I^2 finitely generated as a B-module. By Corollary 2.18, $d^I_{A/R} \otimes_B B_S$ is a split injection, so by Lemma 5.3(2), there exists $s \in S$ such that $d^I_{A/R} \otimes_B B_s$ is a split injection. Applying Corollary 2.18 again, $R \to B_s$ is quasi-smooth. □

To show that in Proposition 5.4, the hypothesis of conormally finite is necessary, let $R^\Lambda = R[X_\lambda]/(X_\lambda^2 - X_\lambda)$, $\mathfrak{a} \subseteq R^\Lambda$ be the ideal generated by the X_λ and $B = R^\Lambda[Y]/\mathfrak{a}Y$, as in Example 4.8, where we saw $R \to B$ is not quasi-smooth. Let $S \subseteq R^\Lambda$ be the multiplicative set of finite products of $X_\lambda - 1$ and fix some $s \in S$. Then $R^\Lambda_S = R^\Lambda/\mathfrak{a}$, and R^Λ_s is the quotient of R^Λ by an ideal generated by finitely many X_λ. Thus as R-algebras, $B_S \cong R[Y]$ and $B_s \cong B$. So $R \to B_S$ is quasi-smooth and $R \to B_s$ is not quasi-smooth. Here, however, by Example 4.8, $\Omega_{B/R}$ is not sum finitely presented. Since all other results on the local nature of quasi-smoothness depend only on $\Omega_{B/R}$ being sum finitely presented, it is natural to ask whether this condition can be substituted for conormally finite in Proposition 5.4. Example 5.6, which will be based on both Example 4.8 and Example 5.5, shows that it can not; in fact, in Example 5.6, the module of differentials is zero.

Example 5.5. Fix a ring R and an integer $\nu \geq 2$. Let $A = R[\sqrt[\nu^n]{Y}] = R[Y, \sqrt[\nu]{Y}, \sqrt[\nu^2]{Y}, \ldots]$. We claim that $R \to A$ is quasi-smooth if and only if ν is nilpotent in R, and that if this is the case, then $\Omega_{A/R} = 0$. Replacing ν by a power of ν does not change A, so in the case ν is nilpotent in R, we may assume $\nu = 0$ in R. Now, letting $R[Y_n] = R[Y_0, Y_1, \ldots]$, we can write $A = R[Y_n]/(Y_n - Y_{n+1}^\nu)$, where $Y_n \mapsto \sqrt[\nu^n]{Y}$. Then the differential map $d^{(Y_n - Y_{n+1}^\nu)}_{R[Y_n]/R}$ sends $Y_n - Y_{n+1}^\nu \mapsto dY_n - \nu Y_{n+1}^{\nu-1}\, dY_{n+1}$. If $\nu = 0$ in R, then $dY_n \mapsto Y_n - Y_{n+1}^\nu$ is an inverse to this map and we conclude that $R \to A$ is quasi-smooth and the cokernel of this map, $\Omega_{A/R}$, is zero.

Suppose now that ν is not nilpotent in R. Let $I^+ = \sum_n \sqrt[\nu^n]{Y} A$ and $S = 1 + I^+ \subseteq A$. We will show that $R \to A_S$ is not quasi-smooth, whence $R \to A$ is not quasi-smooth—we will need the stronger formulation in Example 5.10. Let $\mathfrak{p} \subseteq R$ be a prime ideal not containing ν; it suffices to show that $(R/\mathfrak{p})_\nu \to A_S \otimes_R (R/\mathfrak{p})_\nu$ is not quasi-smooth. Replacing R by $(R/\mathfrak{p})_\nu$, we may assume that ν is a unit in R and that R, and thus also A, is a domain. From the previous paragraph, $\Omega_{A/R}$ is generated by the dY_n, with relations $dY_n = \nu Y_{n+1}^{\nu-1}\, dY_{n+1} = \nu(\sqrt[\nu^{n+1}]{Y})^{\nu-1}\, dY_{n+1}$. Define an A-module map $\Omega_{A/R} \to A$ by $dY_n \mapsto \nu^{-n} \sqrt[\nu^n]{Y}$. Since any element of $\Omega_{A/R}$ is a multiple of a single dY_n, this map is injective. So $\Omega_{A/R}$ is isomorphic to its image $I^+ \subseteq A$, and $\Omega_{A_S/R} \cong I^+_S$. Since I^+_S is a non-finitely generated ideal in a domain, it is not a projective module [Mat2, Theorem 11.3]. Since $\Omega_{A_S/R}$ is not projective, $R \to A_S$ is not quasi-smooth by Corollary 2.19.

Example 5.6. Let R be a ring of characteristic $\nu \geq 2$, and as in Example 4.8, let $R^\Lambda = R[X_\lambda]/(X_\lambda^2 - X_\lambda)$ and $\mathfrak{a} = \sum X_\lambda R^\Lambda$. Now let $A = R^\Lambda[\sqrt[\nu^n]{Y}]$ and $I^+ = \sum_n \sqrt[\nu^n]{Y} A$, as in Example 5.5, so that $R^\Lambda \to A$ is quasi-smooth and $\Omega_{A/R^\Lambda} = 0$. Let $B = A/\mathfrak{a}YA$. Certainly $\Omega_{B/R^\Lambda} = 0$. For any multiplicative set $T \subseteq A$ with image $S \subseteq B$, by Example 2.20, $R^\Lambda \to B_S$ is quasi-smooth iff $(\mathfrak{a}YA)_T = (\mathfrak{a}YA)^2_T$. Since $\mathfrak{a} = \mathfrak{a}^2$ and $Y \in A_T$ is a non-zero divisor, this is true iff $\mathfrak{a}A_T = \mathfrak{a}YA_T$. Select any prime ideal $\mathfrak{p} \subseteq R^\Lambda$ which contains $\mathfrak{a}$, let $\mathfrak{Q} \subseteq A$ be the prime ideal $\mathfrak{p}A + I^+$, and let $\mathfrak{P}$ be the corresponding prime ideal of B. Since $\mathfrak{a}_\mathfrak{p} = 0$ by Example 4.8, we have that $\mathfrak{a}A_\mathfrak{Q} = 0 = \mathfrak{a}YA_\mathfrak{Q}$, so $R^\Lambda \to B_\mathfrak{P}$ is quasi-smooth. For $s \in B - \mathfrak{P}$, select a preimage $t \in A$ and let $\bar{t}$ be its image in $A/I^+ = R^\Lambda$. Since $s \notin \mathfrak{P}$ gives $t \notin \mathfrak{Q} = \mathfrak{p}A + I^+$, we have $\bar{t} \in R^\Lambda - \mathfrak{p}$. Suppose $R^\Lambda \to B_s$ were quasi-smooth, i.e. $\mathfrak{a}A_t = \mathfrak{a}YA_t$. Then in $A_t/I_t^+ = R_{\bar{t}}^\Lambda$, we get $\mathfrak{a}_{\bar{t}} = 0$, which is impossible by Example 4.8. Thus we conclude that $R^\Lambda \to B_s$ is not quasi-smooth.

Here is a result which, like Lemma 5.3, looks at a map from a finitely generated module into a projective one, and therefore will be useful in studying conormally finite maps. For a proof, see [Mat1, Lemma 2 in 29.B], [Mat2, Lemma 2 in Section 28], or [EGA, 0_{IV}.19.1.10].

Proposition 5.7. *Let $g : M \to P$ be a B-module map with M finitely generated and P projective. For an ideal $I \subseteq B$ contained in the Jacobson radical* $\operatorname{rad} B$, *g is a split injection if and only if $g \otimes_B B/I$ is a split injection.*

Lemma 5.8. *Let $R \to B$ be conormally finite, say $R \to A$ is quasi-smooth and $B = A/I$ with I/I^2 finitely generated. Suppose $\mathfrak{a} \subseteq R$ is an ideal such that $\mathfrak{a}B \subseteq$* $\operatorname{rad} B$. *Let "bar" denote reducing modulo $\mathfrak{a}$. Then $R \to B$ is quasi-smooth if and only if $\overline{R} \to \overline{B}$ is quasi-smooth and the canonical surjection $I/I^2 \otimes_R \overline{R} \twoheadrightarrow \overline{I}/\overline{I}^2$ is an isomorphism.*

Proof. The following diagram commutes by Remark 2.15(1).

$$\begin{tikzcd}
I/I^2 \otimes_B \overline{B} \arrow[r, "d^I_{A/R} \otimes_B \overline{B}"] \arrow[d] & (\Omega_{A/R} \otimes_A B) \otimes_B \overline{B} \arrow[d, equal] \\
\overline{I}/\overline{I}^2 \arrow[r, "d^{\overline{I}}_{\overline{A}/\overline{R}}"] & \Omega_{\overline{A}/\overline{R}} \otimes_{\overline{A}} \overline{B}
\end{tikzcd}$$

By Theorem 2.17 and Proposition 5.7, the bottom [resp. top] map is a split injection if and only if $\overline{R} \to \overline{B}$ [resp. $R \to B$] is quasi-smooth. An easy diagram chase completes the lemma. □

Proposition 5.9. *Suppose $R \to B$ is conormally finite and $\mathfrak{a} \subseteq R$ is an ideal such that $\mathfrak{a}B \subseteq$* $\operatorname{rad} B$. *Let "bar" denote reducing modulo $\mathfrak{a}$. If* $\operatorname{Tor}_1^R(B, \overline{R}) = 0$ *and $\overline{R} \to \overline{B}$ is quasi-smooth, then $R \to B$ is quasi-smooth.*

Proof. Apply Lemma 4.5 and Lemma 5.8. □

The following example shows that the condition of conormally finite is necessary in Proposition 5.9. It is open whether we can substitute the condition that $\Omega_{B/R}$ is sum finitely presented.

Example 5.10. Let $p > 0$ be a prime number, R be $\mathbb{Z}$ localized at the prime ideal $p\mathbb{Z}$, $A = R[\sqrt[p^n]{Y}]$, and $I^+ = \sum_n \sqrt[p^n]{Y}A$. By Example 5.5, $R/pR \to A/pA$ is quasi-smooth, and $R \to A_S$ is not quasi-smooth, where $S \subseteq A$ is the multiplicative set $1 + I^+$. Let $\mathfrak{P} \subseteq A$ be the prime ideal generated by p and I^+. Then $A_\mathfrak{P}$ equals A_S, so $R \to A_\mathfrak{P}$ is not quasi-smooth, but $R/pR \to A_\mathfrak{P}/pA_\mathfrak{P}$ is quasi-smooth. Clearly $pA_\mathfrak{P} \subseteq \operatorname{rad} A_\mathfrak{P}$; to see $R \to A_\mathfrak{P}$ is flat (it is, in fact, faithfully flat), notice that A is a free R-module on the basis Y^α, where α ranges over $\{\frac{m}{p^n} \in \mathbb{Q} : m, n \geq 0 \text{ integers}\}$.

Theorem 5.11. *For $R \to B$ conormally finite and flat, the following are equivalent.*

(a) $R \to B$ is quasi-smooth.
(b) For all prime ideals $\mathfrak{p} \subseteq R$, $\kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p})$ is quasi-smooth.
(c) For all maximal ideals $\mathfrak{M} \subseteq B$ with $\mathfrak{p} = \mathfrak{M} \cap R$, $\kappa(\mathfrak{p}) \to B_\mathfrak{M}/\mathfrak{p}B_\mathfrak{M}$ is quasi-smooth.

Proof. (a)⇒(b)⇒(c) is trivial. To show $R \to B$ quasi-smooth, by Theorem 4.7 (and Remark 5.2) it suffices to show that $R_{\mathfrak{p}} \to B_{\mathfrak{M}}$ is quasi-smooth for $\mathfrak{M} \subseteq B$ maximal and $\mathfrak{p} = \mathfrak{M} \cap R$. This follows from (c) by applying Proposition 5.9 with $\mathfrak{a}$ equal to the maximal ideal of $R_{\mathfrak{p}}$. □

Conormally finite is necessary in Theorem 5.11. Indeed, without this hypothesis, $R^{\Lambda} \to B$ in Example 4.8 shows that (b)⇒(a) would not hold. Also $R \to A_S = A_{\mathfrak{P}}$ in Example 5.10 shows (c)⇒(b) would not hold, since $R/pR \to A_{\mathfrak{P}}/pA_{\mathfrak{P}}$ is quasi-smooth, but $\kappa(0) \to A \otimes_R \kappa(0)$, which is $\mathbb{Q} \to \mathbb{Q}[\sqrt[p^n]{Y}]_S$ is not quasi-smooth by Example 5.5. If we only have $\Omega_{B/R}$ sum finitely presented, it is open whether (b)⇒(a), but (c)⇒(b) holds by Theorem 4.7, even without flatness. To see the necessity of flatness in (b)⇒(a), for k a field, consider $k[X]/(X^2) \to k[X]/(X)$, which fails to be quasi-smooth by Example 2.20.

6. The Jacobian Criterion

In light of Section 1, it is no surprise that the Jacobian matrix gives us a tool for determining quasi-smoothness. Approximately speaking, if t is the minimal number of relations over some polynomial ring, then we will have quasi-smoothness if and only if the Jacobian has an invertible $t \times t$ submatrix. The following theorem makes this precise.

Theorem 6.1 (Jacobian Criterion). *Let R be a ring, A a localization of $R[X_\lambda]$ at a prime ideal, $I \subseteq A$ an ideal, and $B = A/I$. Suppose $F_1, \dots, F_t$ in A generate I/I^2. Then the following are equivalent.*

(a) $R \to B$ is quasi-smooth and the F_j give minimal generators of I/I^2.

(b) There exist indices $\lambda_1, \dots, \lambda_t$ such that $\det\left(\frac{\partial F_j}{\partial X_{\lambda_i}}\right)$ *is a unit of A.*

If these conditions hold, then I/I^2 and $\Omega_{B/R}$ are free B-modules with bases $\{F_1, \dots, F_t\}$ and $\{dX_\lambda\}_{\lambda \neq \lambda_1, \dots, \lambda_t}$, respectively.

Remark 6.2. With additional hypotheses, Theorem 7.3(2a) and Theorem 7.14 give that $t = \dim A - \dim B$ and $F_1, \dots, F_t$ is an A-regular sequence. Also see Corollary 7.4.

Proof. By Theorem 2.17 and Example 2.11, $R \to B$ is quasi-smooth iff the map $d = d^I_{A/R}$: $I/I^2 \to \oplus_\lambda B\, dX_\lambda$ is a split injection, where d is induced by $F \mapsto \sum_\lambda \frac{\partial F}{\partial X_\lambda}\, dX_\lambda$.

Assume (b) holds, and consider the following maps, where π is projection: $B^t \overset{(F_j)}{\twoheadrightarrow} I/I^2 \xrightarrow{d} \oplus_\lambda B\, dX_\lambda \xrightarrow{\pi} \oplus_{i=1}^t B\, dX_{\lambda_i}$. The matrix of this composition, when written with respect to the standard basis of B^t and the basis $\{dX_{\lambda_1}, \dots, dX_{\lambda_t}\}$, is the matrix $\left(\frac{\partial F_j}{\partial X_{\lambda_i}}\right)$ after its entries are mapped to B. Since (b) holds, the determinant of this image matrix is a unit of B. Thus the above composition is an isomorphism, which tells us that the first map is an isomorphism, i.e. I/I^2 is free with basis $\{F_j\}$. Also, πd is an isomorphism, which makes $(\pi d)^{-1}\pi$ a splitting map for d, showing that $R \to B$ is quasi-smooth. Now $\Omega_{B/R}$, the cokernel of d, is the kernel of the splitting map. So it is free with basis $\{dX_\lambda\}_{\lambda \neq \lambda_1, \dots, \lambda_t}$, as desired.

Now assume (a), so that d is a split injection. The property of being a split injection is stable under base change, so if L is the residue field of B, $d \otimes_B L : I/I^2 \otimes_B L \to \oplus_\lambda L\, dX_\lambda$ is a split injection. By Nakayama's Lemma, the F_j give a minimal generating set, and thus a basis, of the vector space $I/I^2 \otimes_B L$. Using this basis and the dX_λ, let (m_λ) be the (possibly infinite) matrix for $d \otimes_B L$, each m_λ a $1 \times t$ row vector. Note that all but finitely many m_λ must be zero. Since $I/I^2 \otimes_B L$ is rank t and $d \otimes_B L$ is injective, find indices such that $m_{\lambda_1}, \dots, m_{\lambda_t}$ are linearly independent. Let π be projection $\oplus_\lambda B\, dX_\lambda \twoheadrightarrow \oplus_{i=1}^t B dX_{\lambda_i}$. With respect to the bases of the

F_j and the dX_{λ_i}, $(\pi d) \otimes_B L$ has matrix M with rows $m_{\lambda_1}, \ldots, m_{\lambda_t}$, and is thus an isomorphism. So det M is a unit of L. However the image of $\det\left(\frac{\partial F_j}{\partial X_{\lambda_i}}\right)$ is det M, so $\det\left(\frac{\partial F_j}{\partial X_{\lambda_i}}\right)$ is a unit of A, as desired. □

The following lemma extends Theorem 6.1 to non-local rings, and it will be used to prove Proposition 6.4 and Theorem 6.5.

Lemma 6.3. *Let R be a ring, A a localization of $R[X_\lambda]$ (not necessarily at a prime ideal), $I \subseteq A$ an ideal, and $B = A/I$. Suppose $F_1, \ldots, F_t$ in A generate I/I^2 [resp. generate I]. Then $R \to B$ is quasi-smooth if and only if we have both of the following.*

(a) There exist $G_1, \ldots, G_n$ in $R[X_\lambda]$ generating the unit ideal of B.

(b) For each $1 \le a \le n$, there exist an integer $0 \le t_a \le t$, indices $1 \le \gamma_{a,1}, \ldots, \gamma_{a,t_a} \le t$ such that $F_{\gamma_{a,1}}, \ldots, F_{\gamma_{a,t_a}}$ generate $(I/I^2)_{G_a}$ [resp. generate I_{G_a}] as an A_{G_a}-module, and indices $\lambda_{a,1}, \ldots, \lambda_{a,t_a}$ such that in A, $G_a = H_a \cdot \det\left(\frac{\partial F_{\gamma_{a,j}}}{\partial X_{\lambda_{a,i}}}\right)$ for some $H_a \in R[X_\lambda]$.

If (a) and (b) hold, then for all $1 \le a \le n$, $(I/I^2)_{G_a}$ and $\Omega_{B_{G_a}/R}$ are free B_{G_a}-modules with bases $\{F_{\gamma_{a,1}}, \ldots, F_{\gamma_{a,t_a}}\}$ and $\{dX_\lambda\}_{\lambda \ne \lambda_{a,1}, \ldots, \lambda_{a,t_a}}$, respectively.

Proof. For "if," applying Theorem 6.1 at every prime of B_{G_a}, we get that $R \to B_{\mathfrak{P}}$ is quasi-smooth for all primes $\mathfrak{P} \subseteq B$ not containing the image of G_a. Since this holds for all a, $R \to B$ is quasi-smooth. (Certainly $R \to B$ is conormally finite, and then apply Remark 5.2 and Theorem 4.7.) The last statement of the lemma follows from Theorem 6.1. (Verify that the map from a free module is an isomorphism by checking at all prime ideals.)

For "only if," fix a prime ideal $\mathfrak{P} \subseteq B$ with $\mathfrak{Q} = \mathfrak{P} \cap A$. Select a minimal generating set $F_{\gamma_1}, \ldots, F_{\gamma_{t_{\mathfrak{P}}}}$ of $(I/I^2)_{\mathfrak{Q}}$. If I is finitely generated, then the F_{γ_j} generate $I_{\mathfrak{Q}}$ by Nakayama's Lemma. Since I/I^2 [resp. I] is finitely generated, find $s \in A - \mathfrak{Q}$ such that the F_{γ_j} generate $(I/I^2)_s$ [resp. I_s]. Now, by Theorem 6.1, there exist indices $\lambda_1, \ldots, \lambda_{t_{\mathfrak{P}}}$ such that $D = \det\left(\frac{\partial F_{\gamma_j}}{\partial X_{\lambda_i}}\right) \in A - \mathfrak{Q}$. Writing s and D with numerators and denominators in $R[X_\lambda]$, let G be the product of the numerators of s and D, and H be the numerator of s times the denominator of D, so that $G = HD$ in A. This gives us the desired conclusion in the neighborhood of $\mathfrak{P}$ defined by G. By the quasi-compactness of Spec B, we get the desired result. □

Suppose in Lemma 6.3 that $A = R[X_\lambda]$, I is generated by $F_1, \ldots, F_t$, and (a) and (b) hold; fix $1 \le a \le n$. Note that $D_a = \det\left(\frac{\partial F_{\gamma_j}}{\partial X_{\lambda_i}}\right)$ is a unit of B_{G_a}. Let Λ' be all indices λ except $\lambda_{a,1}, \ldots, \lambda_{a,t_a}$, and let $C \subseteq A$ be the polynomial extension $R[X_\lambda]_{\lambda \in \Lambda'}$. Now $\Omega_{B_{G_a}/R}$ is free with basis $\{dX_\lambda\}_{\lambda \in \Lambda'}$, and so by our choice of C, $\Omega_{B_{G_a}/C} = 0$. Notice $B_{G_a} = C[X_{\lambda_{a,i}}]_{G_a}/(F_{\gamma_{a,j}}) = C[X_{\lambda_{a,i}}, Y]/(F_{\gamma_{a,j}}, 1 - G_a Y)$; this latter formulation, with $t_a + 1$ variables and the same number of relations, has square Jacobian with determinant $D_a G_a$, a unit of B_{G_a}. This makes $C \to B_{G_a}$ quasi-smooth by applying Lemma 6.3. (Indeed, in (a), take $n = 1$ and $G_1 = 1$.) We have shown that a quasi-smooth map $R \to R[X_\lambda]/(F_1, \ldots, F_t)$ locally factors as a polynomial extension followed by an étale map, using the terminology of Remark 3.5. Our conclusions are "only if" in the following proposition, "if" being trivial.

Proposition 6.4. *Let B be a quotient of a polynomial ring over R by a finitely generated ideal. Then $R \to B$ is quasi-smooth if and only if there exist generators G_a of the unit ideal of B such that the following conditions hold for each G_a. The map $R \to B_{G_a}$ factors through some polynomial extension $C = R[X_\lambda]$, where the dX_λ give a basis of the free B_{G_a}-module $\Omega_{B_{G_a}/R}$.*

Further, $C \to B_{G_a}$ is quasi-smooth and $\Omega_{B_{G_a}/C} = 0$; in fact, for some m, B_{G_a} has the form $C[X_1, \ldots, X_m]/(F_1, \ldots, F_m)$, with the square $m \times m$ Jacobian invertible over B_{G_a}.

Using the terminology we will introduce in Section 7, taking a smooth map of finite type $\mathbb{Z}$-algebras and changing base, we clearly get a smooth map; further localizing gives an essentially smooth map. The following theorem, which generalizes [EGA, IV.17.7.9], says, in particular, that all smooth and essentially smooth maps of arbitrary rings arise in this way.

Theorem 6.5. *For elements $F_1, \ldots, F_t$ in $R[X_\lambda]$ and a multiplicative set $S \subseteq R[X_\lambda]$, assume $R \to R[X_\lambda]_S/(F_1, \ldots, F_t)$ is quasi-smooth. Then there exist $s \in S$ and a finite type $\mathbb{Z}$-subalgebra $R_0 \subseteq R$ containing the coefficients of $F_1, \ldots, F_t$, and of s, such that $R_0 \to R_0[X_\lambda]_s/(F_1, \ldots, F_t)$ is quasi-smooth. Further, if S is generated by a finite number of elements, then we can choose s and R_0 such that $R_0[X_\lambda]_s/(F_1, \ldots, F_t) \otimes_{R_0} R = R[X_\lambda]_s/(F_1, \ldots, F_t)$ equals our original ring $R[X_\lambda]_S/(F_1, \ldots, F_t)$.*

Proof. For "further," let s_0 be the product of generators of S. Replace S by $\{s_0^n : n \in \mathbb{N}\}$ and apply the theorem. For the main statement, apply Lemma 6.3 to get n and, for $1 \le a \le n$, an integer t_a, G_a and H_a in $R[X_\lambda]$, and indices $\gamma_{a,1}, \ldots, \gamma_{a,t_a}$, and $\lambda_{a,1}, \ldots, \lambda_{a,t_a}$ which satisfy conditions (a) and (b) of Lemma 6.3. We can rewrite these conditions as follows: there exist elements J_a, K_b, $L_{a,b,1}, \ldots, L_{a,b,t_a}$, in $R[X_\lambda]_S$ (for $1 \le a \le n$ and $1 \le b \le t$) and $N \ge 1$, such that both of the following hold in $R[X_\lambda]_S$.

(a) $\sum_{a=1}^n J_a G_a = 1 + \sum_{b=1}^t K_b F_b$
(b) For $1 \le a \le n$ and $1 \le b \le t$, $G_a^N F_b = \sum_{c=1}^{t_a} L_{a,b,c} F_{\gamma_{a,c}}$, and for $1 \le a \le n$, $G_a = H_a \det\left(\frac{\partial F_{\gamma_{a,j}}}{\partial X_{\lambda_{a,i}}}\right)$.

Let $s_1 \in S$ be a common denominator for all the J_a, K_b, and $L_{a,b,c}$, so that we can consider $s_1 J_a \in R[X_\lambda]$, etc. The above conditions hold if and only if there exists $s_2 \in S$ such that the following hold in $R[X_\lambda]$.

(a′) $\sum_{a=1}^n s_2(s_1 J_a) G_a = s_2 s_1 + \sum_{b=1}^t s_2(s_1 K_b) F_b$
(b′) For $1 \le a \le n$ and $1 \le b \le t$, $s_2 s_1 G_a^N F_b = \sum_{c=1}^{t_a} s_2(s_1 L_{a,b,c}) F_{\gamma_{a,c}}$, and for $1 \le a \le n$, $s_2 G_a = s_2 H_a \det\left(\frac{\partial F_{\gamma_{a,j}}}{\partial X_{\lambda_{a,i}}}\right)$.

Let $R_0 \subseteq R$ be the $\mathbb{Z}$-subalgebra generated by all coefficients appearing in the following elements of $R[X_\lambda]$: s_1, s_2, and all F_b, G_a, H_a, $s_1 J_a$, $s_1 K_b$, and $s_1 L_{a,b,c}$. Let $s = s_1 s_2$. Then, (a′) and (b′) hold in $R_0[X_\lambda]$, so (a) and (b) hold in $R_0[X_\lambda]_s$. By Lemma 6.3, we conclude $R_0 \to R_0[X_\lambda]_s/(F_1, \ldots, F_t)$ is quasi-smooth. □

The remainder of this section will not be used in the later sections. It shows that $H_{B/R}$, which gives the complement of the quasi-smooth locus, should be defined in the generality of conormally finite maps, or at least for maps satisfying the conditions of Proposition 6.10 below.

Definition 6.6. *For $R \to B$ conormally finite, let $H_{B/R} = \{b \in B : R \to B_b$ is quasi-smooth$\}$.*

Proposition 6.7. *Suppose $R \to B$ is a conormally finite map.*

1. *Let $\mathcal{X} = \{$primes $\mathfrak{P} \subseteq B : R \to B_{\mathfrak{P}}$ is not quasi-smooth$\}$. Then $H_{B/R} = \cap_{\mathfrak{P} \in \mathcal{X}} \mathfrak{P}$. In particular, $H_{B/R}$ is a radical ideal.*
2. *For $S \subseteq B$ a multiplicative set, $H_{B_S/R} = (H_{B/R})_S$.*
3. *For $\mathfrak{P} \subseteq B$ prime, $R \to B_{\mathfrak{P}}$ is quasi-smooth if and only if $\mathfrak{P} \not\supseteq H_{B/R}$.*

Proof. Using Theorem 4.7, we have $b \in H_{B/R}$ iff for all prime ideals $\mathfrak{P} \subseteq B$ not containing b, $R \to B_{\mathfrak{P}}$ is quasi-smooth iff for all $\mathfrak{P} \in \mathcal{X}$, $b \in \mathfrak{P}$. Thus we have (1). Note that (2) follows from (1).

For (3), if $\mathfrak{P} \not\supseteq H_{B/R}$, then $\mathfrak{P} \notin \mathcal{X}$ by (1), so $R \to B_{\mathfrak{P}}$ is quasi-smooth. Conversely, if $R \to B_{\mathfrak{P}}$ is quasi-smooth, then using Proposition 5.4, there exists $b \in B - \mathfrak{P}$ such that $R \to B_b$ is quasi-smooth, i.e. $b \in H_{B/R} - \mathfrak{P}$. □

Remark 6.8. By the proof of Proposition 6.7, (1) and (2) hold if we weaken the hypothesis of $R \to B$ conormally finite to $\Omega_{B/R}$ sum finitely presented. However, with such a weakening, Example 5.6 shows that (3) fails to hold.

Definition 6.9. *For $F_1, \ldots, F_t$ in $R[X_\lambda]$, let $\Delta(F_1, \ldots, F_t) \subseteq R[X_\lambda]$ be the ideal generated by the $t \times t$ minors of the (possibly infinite) matrix $\left(\frac{\partial F_j}{\partial X_\lambda}\right)$. By convention, the determinant of the 0×0 matrix is 1, so that if $t = 0$, $\Delta() = R[X_\lambda]$. Also, if the number of variables X_λ is finite and less than t, we set $\Delta(F_1, \ldots, F_t) = 0$.*

Suppose $B = R[X_\lambda]/I$. For any $f_1, \ldots, f_t$ in I/I^2, select preimages $F_1, \ldots, F_t$ in I. Let $\Delta'(f_1, \ldots, f_t) \subseteq B$ be the ideal $\Delta(F_1, \ldots, F_t)B$. Note that $\Delta'(f_1, \ldots, f_t)$ is independent of the choice of the F_j.

Proposition 6.10. *Let $B = R[X_\lambda]/I$.*

(1) If I/I^2 is a finitely generated B-module, then we have the following.

$$H_{B/R} = \sqrt{\sum_{t \geq 0,\ \{f_1, \ldots, f_t\} \subseteq I/I^2} [(f_1, \ldots, f_t) : I/I^2]\ \Delta'(f_1, \ldots, f_t)}$$

(2) If I is a finitely generated A-module, then we have the following.

$$H_{B/R} = \sqrt{\sum_{t \geq 0,\ \{F_1, \ldots, F_t\} \subseteq I} [(F_1, \ldots, F_t) : I]\ \Delta(F_1, \ldots, F_t)B}$$

Proof. Let the ideal $J \subseteq B$ be the right hand side of the equation in (1). Since by Proposition 6.7(1) $H_{B/R}$ is also a radical ideal, by Proposition 6.7(3), it suffices to show that for a prime ideal $\mathfrak{P} \subseteq B$, $\mathfrak{P} \not\supseteq J$ iff $R \to B_{\mathfrak{P}}$ is quasi-smooth. Now $\mathfrak{P} \not\supseteq J$ iff for some $f_1, \ldots, f_t$ in I/I^2, $\mathfrak{P} \not\supseteq [(f_1, \ldots, f_t) : I/I^2]$ and $\mathfrak{P} \not\supseteq \Delta'(f_1, \ldots, f_t)$. If $\mathfrak{Q} = \mathfrak{P} \cap R[X_\lambda]$, this is equivalent to having some $F_1, \ldots, F_t$ in I, $F_1, \ldots, F_t$ generating $(I/I^2)_{\mathfrak{Q}}$, with $\mathfrak{Q} \not\supseteq \Delta(F_1, \ldots, F_t)$, i.e. $R \to B_{\mathfrak{P}}$ is quasi-smooth, by Theorem 6.1. (2) is proved similarly, because Nakayama's Lemma gives us that $F_1, \ldots, F_t$ generate $I_{\mathfrak{Q}}$ iff they generate $(I/I^2)_{\mathfrak{Q}}$. □

Note that $H_{B_S/R}$ can be calculated by using Proposition 6.7(2) and Proposition 6.10.

7. Essential Smoothness

In this section we will look at how quasi-smoothness relates to flatness, dimension, regular rings, and regular sequences. Except for Corollary 7.8, we can not get our desired results by looking at localizations of finite type maps. We therefore make the following definitions, again following the terminology used by Swan [Sw].

Definition 7.1. *Recall that a map of rings $R \to B$ is* finitely presented *if we can express $B = A/I$ where $A = R[X_i]$ is a polynomial ring in finitely many variables and I is a finitely generated ideal; we say $R \to B$ is* essentially finitely presented *if instead we only require A to be a localization of $R[X_i]$, or equivalently, that B is a localization of a finitely presented R-algebra.*

We say a map of rings is smooth *if it is quasi-smooth and finitely presented, and is* essentially smooth *if it is quasi-smooth and essentially finitely presented.*

Note that if $R \to B$ is essentially finitely presented, then it is conormally finite, and also $\Omega_{B/R}$ is a finitely presented B-module. We begin by recalling those properties about dimension which we will use.

Proposition 7.2.

(1) Let $R \to B$ be a flat map of Noetherian rings. If $\mathfrak{P} \subseteq B$ is prime with $\mathfrak{p} = \mathfrak{P} \cap R$, then $\dim B_{\mathfrak{P}} = \dim R_{\mathfrak{p}} + \dim B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$.

(2) If B is a finite type domain over a field and $\mathfrak{P} \subseteq B$ is prime, then $\dim B = \dim B_{\mathfrak{P}} + \dim B/\mathfrak{P}$.

(3) If B is a finite type algebra over a field k and $\mathfrak{P} \subseteq B$ is prime, then $\dim B/\mathfrak{P} = \text{tr. deg.}_k\, \kappa(\mathfrak{P})$. *In particular, if $\mathfrak{M} \subseteq B$ is maximal, then $k \subseteq B/\mathfrak{M}$ is an algebraic field extension.*

(4) For R Noetherian, $\mathfrak{Q} \subseteq R[X_1, \ldots, X_s]$ a prime ideal, and $\mathfrak{p} = \mathfrak{Q} \cap R$, we have that $\dim R[X_1, \ldots, X_s]_{\mathfrak{Q}} = \dim R_{\mathfrak{p}} + s - \text{tr. deg.}_{\kappa(\mathfrak{p})}\, \kappa(\mathfrak{Q})$.

Proof. (1) can be found in [Mat1, 13.B] or [Mat2, Theorem 15.1], (2) in [Mat1, 14.H], and (3) follows from Noether normalization [Mat1, 14.G]. (4) is proved in [Mat1, 14.C], or can be concluded from (1), (2), and (3). □

In Theorem 7.3, we gather together various results we need. We immediately prove most of the theorem—in particular, all parts will be established for R regular—and will complete the proof at the end of this section.

Theorem 7.3. *For $R \to B$ an essentially finitely presented local map of local rings, let k and L be the residue fields of R and B, respectively.*

(1) a. If R is Noetherian, then $\dim B \leq \dim R + \text{rank}_L\, \Omega_{B/R} \otimes_B L - \text{tr. deg.}_k\, L$*; if also $R \to B$ is essentially smooth, then equality holds.*

b. If R is regular local, then $R \to B$ is essentially smooth if and only if $\dim B = \dim R + \text{rank}_L\, \Omega_{B/R} \otimes_B L - \text{tr. deg.}_k\, L$.

c. If R is regular local and $R \to B$ is essentially smooth, then B is regular local.

Fix a presentation $B = A/(F_1, \ldots, F_t)$, taking A to be a localization of $R[X_1, \ldots, X_s]$ at a prime ideal, and let r be the rank of the $s \times t$ matrix M formed by taking the Jacobian matrix $\left(\frac{\partial F_j}{\partial X_i}\right)$ and mapping its entries to L.

(2) a. Suppose A is Noetherian. Then $r \leq \dim A - \dim B \leq t$. If also $F_1, \ldots, F_t$ are minimally chosen, then $R \to B$ is essentially smooth if and only if $r = \dim A - \dim B = t$.

b. If R is regular local, then $R \to B$ is essentially smooth if and only if $r = \dim A - \dim B$.

c. $r = s - \text{rank}_L\, \Omega_{B/R} \otimes_B L$.

Let $B_0 = A/(F_{j_1}, \ldots, F_{j_r})$, where r is as above and $F_{j_1}, \ldots, F_{j_r}$ are chosen so that the corresponding columns of M are linearly independent.

(3) a. $R \to B_0$ is essentially smooth.

b. $R \to B$ is essentially smooth if and only if $B = B_0$.

c. If A is Noetherian, then $\dim B_0 = \dim A - r$.

d. If R is Noetherian, then $\dim B_0 = \dim R + \text{rank}_L\, \Omega_{B/R} \otimes_B L - \text{tr. deg.}_k\, L$.

e. If R is regular local, then so is B_0.

Partial Proof of Theorem 7.3. Let $I \subseteq A$ be generated by $F_1, \dots, F_t$. For (2c), take the composition $B^t \overset{(F_j)}{\twoheadrightarrow} I/I^2 \overset{d^I_{A/R}}{\to} \Omega_{A/R} \otimes_A B$ and change base to the residue field L. The resulting map has matrix M, when written with respect to the standard basis of B^t and $dX_1, \dots, dX_s$. Therefore its cokernel, $\Omega_{B/R} \otimes_B L$, has rank $s - r$.

Let $\mathfrak{m}_A$ be the maximal ideal of A. We claim the following: for any ideal $I' \subseteq \mathfrak{m}_A$, if $F_{j_1}, \dots, F_{j_r}$ are contained in I', then they give linearly independent elements of the L-vector space $I'/\mathfrak{m}_A I'$. Indeed, otherwise for some $a \leq r$, we would have $F_{j_a} \in \mathfrak{m}_A I' + \sum_{k \neq a} F_{j_k} A$. Using $\frac{\partial(G_1 G_2)}{\partial X_i} = G_1 \frac{\partial G_2}{\partial X_i} + G_2 \frac{\partial G_1}{\partial X_i}$ we could conclude that the j_a-th column of the matrix M is a linear combination of the other j_k-th columns, $k \neq a$, which is a contradiction.

The Jacobian Criterion, Theorem 6.1, gives us (3a), which in turn gives half of (3b); for the other half, suppose $R \to B$ is essentially smooth. By the claim, $F_{j_1}, \dots, F_{j_r}$ give linearly independent elements of $I/\mathfrak{m}_A I$. Expand the set to $F_{j_1}, \dots, F_{j_n}$ which give a basis of $I/\mathfrak{m}_A I$, for some $n \geq r$. By Nakayama's Lemma, $F_{j_1}, \dots, F_{j_n}$ give a minimal generating set of both I and I/I^2. Since $R \to B$ is essentially smooth, the Jacobian Criterion tells us that the rank of the matrix M is n, so $r = n$. But since $F_{j_1}, \dots, F_{j_n}$ generate I, we conclude $B = B_0$.

We will now show (3e). By the claim above, $F_{j_1}, \dots, F_{j_r}$ give linearly independent elements of $\mathfrak{m}_A/\mathfrak{m}_A^2$. If R, and thus also A, is regular local, this tells us that B_0 is a regular local ring of dimension $\dim A - r$ [Mat2, Theorem 14.2], which completes (3e). (1c) follows from (3b) and (3e).

We have also just shown (3c) for R regular local. We will prove the general case at the end of this section, following Theorem 7.14. For now we assume (3c) and finish the theorem. This will establish all parts of the theorem for R regular local and leave only (3c) unfinished for later.

For R Noetherian, we get (3d) by combining (3c), (2c), and $\dim A = \dim R + s - \operatorname{tr.deg.}_k L$, which comes from Proposition 7.2(4).

To complete (1) and (2), we make the following observations. For A Noetherian, since B is a quotient of B_0, $\dim B \leq \dim B_0$. We have equality if $R \to B$ is essentially smooth, by (3b). For R regular local, equality is equivalent to $R \to B$ being essentially smooth. Indeed, additionally use that $B = B_0$ iff $\dim B = \dim B_0$, since B_0 is a domain by (3e).

For R Noetherian, the previous paragraph gives (1a) and (1b) by using (3d) to calculate $\dim B_0$. For A Noetherian instead, if we use (3c) to calculate $\dim B_0$, we see $\dim B \leq \dim B_0$ is equivalent to $r \leq \dim A - \dim B$. This gives (2b) and the first inequality of (2a). To complete (2a), notice that $\dim A - \dim B \leq t$ is standard dimension theory, e.g. use [Mat1, 12.H] or [Mat2, Theorem 13.4]. And for $F_1, \dots, F_t$ minimally chosen, use (3b) to conclude that $R \to B$ is essentially smooth iff $r = t$. □

The condition that R is regular local is certainly necessary to conclude essential smoothness in Theorem 7.3(1b),(2b). Indeed, for $k = L$ a field, let $R = A = k[X]/(X^2)$ and $B = R/(X)$. Then $0 = r = \dim A = \dim B = \dim R = \operatorname{rank}_L \Omega_{B/R} \otimes_B L = \operatorname{tr.deg.}_k L$, but $R \to B$ is not essentially smooth by Example 2.20.

The following corollary links essential smoothness to the older concept of absolutely simple points, which we discussed in Section 1.

Corollary 7.4. *Let k be a field, $B = k[X_1, \dots, X_s]/(F_1, \dots, F_t)$ a domain, and $\mathfrak{P} \subseteq B$ a prime ideal. Then $k \to B_{\mathfrak{P}}$ is essentially smooth if and only if the Jacobian matrix $\left(\frac{\partial F_j}{\partial X_i}\right)$, when its entries are mapped to $\kappa(\mathfrak{P})$, has rank $s - \dim B$.*

Proof. Let $A = k[X_1, \dots, X_s]$ and $\mathfrak{Q} = \mathfrak{P} \cap A$. Applying Theorem 7.3(2b) to $k \to B_{\mathfrak{P}}$, it suffices to show that $\dim A_{\mathfrak{Q}} - \dim B_{\mathfrak{P}} = s - \dim B$. By Proposition 7.2(2), $\dim A_{\mathfrak{Q}} - \dim B_{\mathfrak{P}} = (\dim A - \dim A/\mathfrak{Q}) - (\dim B - \dim B/\mathfrak{P})$. Since $A/\mathfrak{Q} = B/\mathfrak{P}$, this equals $s - \dim B$. □

Corollary 7.5. *For R a regular ring, let $R \to B$ be essentially finitely presented.*

(1) *If $R \to B$ is essentially smooth, then B is a regular ring. Conversely, if B is a regular ring and $R \to B/\mathfrak{M}$ is essentially smooth for all maximal ideals $\mathfrak{M} \subseteq B$, then $R \to B$ is essentially smooth.*

(2) *Suppose B is local and $b_1, \dots, b_n$ are in the maximal ideal of B. If $R \to B/(b_1, \dots, b_n)$ is essentially smooth and* $\dim B/(b_1, \dots, b_n) = \dim B - n$, *then $R \to B$ is essentially smooth.*

Proof. For (2), use Proposition 2.4 to replace R by a regular local ring. Choose a presentation $B = A/(F_1, \dots, F_t)$ with A a localization of $R[X_1, \dots, X_s]$ at a prime ideal, and choose $F_{t+1}, \dots, F_{t+n}$ in A mapping to $b_1, \dots, b_n$. Let r be the rank of the $s \times t$ matrix M formed by taking the Jacobian for $F_1, \dots, F_t$ and mapping its entries to the residue field of B; similarly choose r' and M' for $F_1, \dots, F_{t+n}$. Using Theorem 7.3(2b) $r' = \dim A - \dim B/(b_1, \dots, b_n) = \dim A - \dim B + n$. Since M is a submatrix of M' with n fewer columns, $r \geq r' - n = \dim A - \dim B$. Applying Theorem 7.3(2a),(2b), $r = \dim A - \dim B$ and $R \to B$ is essentially smooth.

The first part of (1) follows from Theorem 7.3(1c) (and Proposition 2.4). For the converse, by Theorem 4.7 we may assume B is regular local, so choose $b_1, \dots, b_{\dim B}$ which generate the maximal ideal, and then apply (2). □

Proposition 7.6.

(1) *A field extension is essentially smooth if and only if it is separably generated. In particular, a finitely generated field extension of a perfect field is essentially smooth.*

(2) *Suppose k is a perfect field and B is any localization of a finite type k-algebra. Then $k \to B$ is essentially smooth if and only if B is a regular ring.*

(3) *Suppose k is a field and B is any localization of a finite type k-algebra. Then $\Omega_{B/k} = 0$ if and only if B is a finite product of finite separable field extensions of k. If this is the case, then $k \to B$ is essentially smooth.*

Remark 7.7. Generalizing (1), a field extension is quasi-smooth if and only if it is separable. See [EGA, 0_{IV}.19.6.1] or [Sw, pp. 149–150].

Proof. Note that a field extension is finitely generated iff it is an essentially finitely presented ring map. For (1), [Mat1, Theorem 59(iii)] gives that a finitely generated field extension $k \subseteq L$ is separably generated iff $\operatorname{rank}_L \Omega_{L/k} = \text{tr. deg.}_k L$. By Theorem 7.3(1b), this is equivalent to $k \to L$ being essentially smooth.

Each condition of (2) and (3) can be checked at all maximal ideals of B. (Use Theorem 4.7.) So assume B is local, say with maximal ideal $\mathfrak{M}$. Then (2) follows from Corollary 7.5(1) and (1), above. For (3), if $\Omega_{B/k} = 0$, then by using all three parts of Theorem 7.3(1), $\dim B = 0 = \text{tr. deg.}_k B/\mathfrak{M}$, $k \to B$ is essentially smooth, and B is regular local, thus a domain. So $B = B/\mathfrak{M}$ is a finite field extension of k, with separability by (1). Conversely, if B is a finite separable field extension of k, then by [Mat1, Theorem 59(iii)] (quoted above), $\operatorname{rank}_B \Omega_{B/k} = \text{tr. deg.}_k B = 0$. □

Corollary 7.8. *Suppose B is any localization of a finite type R-algebra and $\Omega_{B/R} = 0$. If $R \to B$ is conormally finite and flat, then it is quasi-smooth.*

Proof. By Theorem 5.11, it suffices to show $\kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p})$ is quasi-smooth for all prime ideals $\mathfrak{p} \subseteq R$, but this follows from Proposition 7.6(3). □

As we see in the following example, neither Proposition 7.6(3) (taking, below, R to be a field) nor Corollary 7.8 holds if we remove the hypothesis that B is a localization of a finite type algebra, even if $R \to B$ is conormally finite.

Example 7.9. Let R be a ring of characteristic $\nu \geq 2$, $A = R[\sqrt[\nu^n]{Y}]$, and $B = A/YA$. By Example 5.5, $\Omega_{A/R} = 0$ (so $\Omega_{B/R} = 0$) and $R \to A$ is quasi-smooth. Since YA is finitely generated, $R \to B$ is conormally finite. Also, $YA \neq (YA)^2$, so by Example 2.20, $R \to B$ is not quasi-smooth. To see that $R \to B$ is faithfully flat, notice that B is a free R-module with basis Y^α, where α ranges over $\{\frac{m}{\nu^n} \in \mathbb{Q} : m \geq 0, n > 0 \text{ integers, and } 0 \leq \frac{m}{\nu^n} < 1\}$.

Lemma 7.10. *Let $R \to A$ be essentially smooth and a local map of local rings. Suppose $I \subseteq A$ is an ideal such that $R \to A/I$ is also essentially smooth. Then any minimal generating set of I maps to a regular sequence in $\overline{A}$, where "bar" denotes reducing modulo the maximal ideal of R.*

Proof. Since $R \to A$ and $R \to A/I$ are essentially finitely presented, I is finitely generated. By Nakayama's Lemma, we prove this lemma for $y_1, \dots, y_n$ which give a minimal generating set of I/I^2, rather than of I.

Applying Nakayama's Lemma again, $y_1, \dots, y_n$ form a minimal generating set of $I/I^2 \otimes_R \overline{R}$, which is isomorphic to $\overline{I}/\overline{I}^2$ by Lemma 5.8. So replacing R, A, and I by $\overline{R}$, $\overline{A}$, and $\overline{I}$, we may assume further that $R = k$ is a field.

By Theorem 2.17, $(\Omega_{A/k} \otimes_A A/I) \cong I/I^2 \oplus \Omega_{(A/I)/k}$, which is projective by Corollary 2.19. Since A/I is local, we have that I/I^2 is free and $y_1, \dots, y_n$ give a basis, being a minimal generating set. Letting L be the common residue field of A and A/I, we conclude that $\operatorname{rank}_L \Omega_{A/k} \otimes_A L = n + \operatorname{rank}_L \Omega_{(A/I)/k} \otimes_{A/I} L$. This calculation, after applying Theorem 7.3(1c),(1b) with $B = A$ and then with $B = A/I$, tells us that A and A/I are regular local rings and that $\dim A = n + \dim A/I$. Thus, since $y_1, \dots, y_n$ generate I, they are part of a regular system of parameters for A, and thus form a regular sequence [Mat2, Theorem 14.2 and Theorem 17.4]. □

Here we recall [Mat2, Corollary to Theorem 22.5] which is proved using the local criteria of flatness. A similar result, which could be used in our applications instead, is [EGA, IV.11.3.8].

Proposition 7.11. *Let $R \to A$ be a local map of Noetherian local rings, and $y_1, \dots, y_n$ be elements of the maximal ideal of A. Let "bar" denote reducing modulo the maximal ideal of R. Then the following are equivalent.*

(a) $y_1, \dots, y_n$ is an A-regular sequence and $R \to A/(y_1, \dots, y_n)$ is flat.
(b) $\overline{y}_1, \dots, \overline{y}_n$ is an $\overline{A}$-regular sequence and $R \to A$ is flat.

Theorem 7.12. *Let $R \to B$ be essentially finitely presented.*

(1) $R \to B$ is essentially smooth if and only if it is flat and for all prime ideals $\mathfrak{p} \subseteq R$, $\kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p})$ is essentially smooth.
(2) If $\Omega_{B/R} = 0$, then $R \to B$ is essentially smooth if and only if it is flat.

Proof. For (1), we have "if" by Theorem 5.11, and in "only if," clearly any $\kappa(\mathfrak{p}) \to B \otimes_R \kappa(\mathfrak{p})$ is essentially smooth. So we need to show that an essentially smooth map $R \to B$ is flat.

First assume $R \to B$ is essentially smooth with R Noetherian. Write $B = A/I$ with $A = R[X_i]_S$ and I finitely generated. Let $\mathfrak{P} \subseteq B$ be prime, $\mathfrak{Q} = \mathfrak{P} \cap A$, and $\mathfrak{p} = \mathfrak{Q} \cap R$. Applying Lemma 7.10 to $R_\mathfrak{p} \to A_\mathfrak{Q}$ and $I_\mathfrak{Q}$, we get that $I_\mathfrak{Q}$ is generated by elements $y_1, \ldots, y_n$ such that $\overline{y}_1, \ldots, \overline{y}_n$ form an $\overline{A}_\mathfrak{Q}$-regular sequence, where "bar" denotes reducing modulo the maximal ideal of $R_\mathfrak{p}$. Now $R_\mathfrak{p} \to A_\mathfrak{Q}$ is flat, so by Proposition 7.11, $R_\mathfrak{p} \to B_\mathfrak{P}$ is flat. Since this holds for all primes $\mathfrak{P} \subseteq B$, $R \to B$ is flat.

For an arbitrary essentially smooth map $R \to B$, by Theorem 6.5, find a finite type $\mathbb{Z}$-subalgebra $R_0 \subseteq R$ and a smooth map $R_0 \to B_0$ such that B is a localization of $B \otimes_{R_0} R$. As we saw above, $R_0 \to B_0$ is flat, and thus so is $R \to B$. This completes (1). (2) is immediate from (1) and Corollary 7.8. □

Using the terminology in Remark 3.5, (2) in the preceding theorem says that an essentially finitely presented map is étale if and only if it is flat and unramified. In (1), note that a quasi-smooth map may not be flat if it is not essentially finitely presented, even if it is conormally finite, which we see in the following example.

Example 7.13. Let R be any non-zero ring, $\nu \geq 2$ an integer, $A = R[\sqrt[\nu^\infty]{Y}]$, and $I^+ = \sum_n \sqrt[\nu^n]{Y} A$, as in Example 5.5. Since $I^+ = (I^+)^2$, we get that $A \to A/I^+$ is conormally finite, and it is quasi-smooth by Example 2.20. However, $A \to A/I^+$ is not flat since multiplication by Y is injective on A but zero on A/I^+.

The following theorem, which extends [EGA, IV.17.12.1], relates essential smoothness to local complete intersection ideals.

Theorem 7.14. *Let $R \to A$ be essentially finitely presented, $I \subseteq A$ a finitely generated ideal, and $\mathfrak{Q} \subseteq A$ a prime ideal containing I such that $R \to A_\mathfrak{Q}/I_\mathfrak{Q}$ is essentially smooth. Fix $y_1, \ldots, y_n$ in A which give a minimal generating set of $I_\mathfrak{Q}$. Then the following are equivalent.*

(a) $R \to A_\mathfrak{Q}$ is essentially smooth.

(b) There exists $s \in A - \mathfrak{Q}$ such that $R \to A_s$ is essentially smooth.

(c) $y_1, \ldots, y_n$ gives an $A_\mathfrak{Q}$-regular sequence.

(d) There exists $s \in A - \mathfrak{Q}$ such that $y_1, \ldots, y_n$ gives an A_s-regular sequence generating I_s.

Remark 7.15. Suppose we exchange the hypothesis that $R \to A$ is essentially finitely presented with the hypothesis that A is Noetherian. Then (c)⇔(d) (see proof below), and by using [EGA, 0_{IV}.19.5.4 and 0_{IV}.15.1.11], if $R \to A_\mathfrak{Q}$ is quasi-smooth, then (c) holds.

Lemma 7.16. *Let A_0 and R be R_0-algebras and $A = A_0 \otimes_{R_0} R$. For $y \in A_0$, suppose $\operatorname{Tor}_1^{R_0}(A_0/yA_0, R) = 0$. Then $\operatorname{Ann}_{A_0} y$ generates the ideal $\operatorname{Ann}_A y$. In particular, if y is A_0-regular, then y is also A-regular. (We abusively use y also for its image in A.)*

Proof. By Lemma 4.5, $yA_0 \otimes_{R_0} R \cong yA$. Thus taking the exact sequence $\operatorname{Ann}_{A_0} y \to A_0 \xrightarrow{y} yA_0 \to 0$ and tensoring over R_0 with R, we get the exact sequence $(\operatorname{Ann}_{A_0} y) \otimes_{R_0} R \to A \xrightarrow{y} yA \to 0$. □

Proof of Theorem 7.14. (b)⇒(a) and (d)⇒(c) are trivial, and (a)⇒(b) follows from Proposition 5.4. Throughout this proof we will use, without explicitly citing Theorem 7.12(1), that essentially smooth maps are flat.

Suppose A is Noetherian. Find $s_0 \in A - \mathfrak{Q}$ such that $y_1, \ldots, y_n$ generate I_{s_0}. By considering the finitely generated kernels of the endomorphisms on $A/(y_1, \ldots, y_{a-1})$, $1 \leq a \leq n$, given by multiplication by y_a, we get (c)⇒(d) for A Noetherian.

Let us prove the theorem for R Noetherian. We already have (a)⇔(b) and (c)⇔(d). If $\mathfrak{p} = \mathfrak{Q} \cap R$, then showing (a)⇔(c) for $R \to A$, I, and $\mathfrak{Q}$ is equivalent to doing so for $R_\mathfrak{p} \to A_\mathfrak{Q}$,

$I_{\mathfrak{Q}}$, and $\Omega_{\mathfrak{Q}}$, using Proposition 2.4. So without loss of generality, we may assume $R \to A$ is a local map of Noetherian local rings, $\mathfrak{Q}$ is the maximal ideal of A, $R \to A/I$ is essentially smooth, and $y_1, \ldots, y_n$ give a minimal generating set of I. Let "bar" denote reducing modulo the maximal ideal of R.

Suppose (a) holds, i.e. $R \to A$ is essentially smooth, and thus flat. By Lemma 7.10, $\overline{y}_1, \ldots, \overline{y}_n$ is an $\overline{A}$-regular sequence. So applying Proposition 7.11, we get that $y_1, \ldots, y_n$ is an A-regular sequence. Conversely, if (c) holds, then since $R \to A/I$ is flat (being essentially smooth), apply Proposition 7.11 to conclude that $R \to A$ is flat and $\overline{y}_1, \ldots, \overline{y}_n$ is an $\overline{A}$-regular sequence. In particular, $\dim \overline{A}/\overline{I} = \dim \overline{A} - n$, and so by Corollary 7.5(2), $\overline{R} \to \overline{A}$ is essentially smooth. Thus $R \to A$ is essentially smooth by Proposition 5.9. We have proved the theorem for R Noetherian.

Let us now prove (a)$\Rightarrow$(d). Write $A = R[X_i]_S/(F_j)$ with finitely many F_j in $R[X_i]$, and let G_k in $R[X_i]$ map to a finite generating set of I. With $\mathfrak{Q}' = \mathfrak{Q} \cap R[X_i]$, by Theorem 6.5 find $s_1 \in R[X_i] - \mathfrak{Q}'$ and a finite type $\mathbb{Z}$-subalgebra $R_0 \subseteq R$ containing the coefficients of s_1, the F_j, and the G_k such that if $A_0 = R_0[X_i]/(F_j)$ and $I_0 = \sum_k G_k A_0$, then we have that $R_0 \to (A_0)_{s_1}$ and $R_0 \to (A_0/I_0)_{s_1}$ are both smooth. Indeed, apply Theorem 6.5 twice, letting s_1 be the product of the elements found and R_0 be the composite of the subalgebras. Changing $y_1, \ldots y_n$ by units of A, we may assume they are in $R[X_i]/(F_j)$; further enlarging R_0, we may assume they are in A_0 and that they give a minimal generating set of $(I_0)_{\mathfrak{Q} \cap A}$. Indeed, take equations showing that they generate the G_k in $A_{\mathfrak{Q}}$, and enlarge R_0 by adding any elements of R which appear in the equations.

Since R_0 is Noetherian, apply (a)$\Rightarrow$(d) to $R_0 \to A_0$, I_0, and $\mathfrak{Q} \cap A_0$ to get $s_2 \in A_0 - \mathfrak{Q} \cap A_0$ such that $y_1, \ldots, y_n$ is an $(A_0)_{s_2}$-regular sequence generating $(I_0)_{s_2}$. Let $s = s_1 s_2$. Then $y_1, \ldots, y_n$ is an $(A_0)_s$-regular sequence which generates $(I_0)_s$ (and thus also I_s); also $R_0 \to (A_0/I_0)_s$ is smooth. Now for $1 \leq a \leq n$, each $R_0 \to (A_0)_s/(y_1, \ldots, y_a)$ is flat, because with R_0 Noetherian we apply (d)$\Rightarrow$(a) for the regular sequence $y_{a+1}, \ldots, y_n$ to conclude that every localization of $(A_0)_s/(y_1, \ldots, y_a)$ at a prime ideal is an essentially smooth, and thus flat, R_0-algebra. Using this flatness, repeatedly apply the last statement of Lemma 7.16 to conclude $y_1, \ldots, y_n$ is an $(A_0 \otimes_{R_0} R)_s$-regular sequence. It is thus also an A_s-regular sequence since A is a localization of $A_0 \otimes_{R_0} R$.

Finally we show (c)$\Rightarrow$(a). Replacing A by $A_{\mathfrak{Q}}$, we may assume A is local with maximal ideal $\mathfrak{Q}$, and I is generated by the A-regular sequence $y_1, \ldots, y_n$. By proving one by one that $R \to A/(y_1, \ldots, y_a)$ is essentially smooth for $a = n-1, n-2, \ldots, 0$, we can assume I is generated by a single A-regular element y.

Write $A = R[X_i]_{\mathfrak{Q}'}/(F_j)$ with finitely many F_j in $R[X_i]$, and let $G \in R[X_i]_{\mathfrak{Q}'}$ map to y, so that $A/I = R[X_i]_{\mathfrak{Q}'}/(F_j, G)$. Changing y by a unit of A, we may assume $G \in R[X_i]$. By Theorem 6.5, find $s \in R[X_i] - \mathfrak{Q}'$ and a finite type $\mathbb{Z}$-subalgebra $R_0 \subseteq R$ containing the coefficients of s, G, and the F_j such that letting $A_0 = R_0[X_i]_s/(F_j)$, we have that $R_0 \to A_0/yA_0$ is smooth.

Using that A_0 is Noetherian, find a finite number of elements $H_k \in R_0[X_i]$ which map to generators of the ideal $\mathrm{Ann}_{A_0} y$. Since y is A-regular, the H_k map to zero in A. Thus, in $R[X_i]$, $MH_k = \sum_j N_{jk} F_j$ for some $M \in R[X_i] - \mathfrak{Q}'$, and some $N_{jk} \in R[X_i]$. Let R_1 be the subring of R generated by R_0 and the coefficients of M and the N_{jk}; let $A_1 = R_1[X_i]_s/(F_j)$ and $\mathfrak{Q}_1 = \mathfrak{Q} \cap A_1$. Notice that $\mathrm{Ann}_{A_0} y$ maps to zero in $(A_1)_{\mathfrak{Q}_1}$.

Now $R_0 \to A_0/yA_0$ is smooth, and thus flat. So applying Lemma 7.16, $\mathrm{Ann}_{A_0} y$ generates $\mathrm{Ann}_{A_1} y$, and thus it also generates $(\mathrm{Ann}_{A_1} y)_{\mathfrak{Q}_1} = \mathrm{Ann}_{(A_1)_{\mathfrak{Q}_1}} y$. But since $\mathrm{Ann}_{A_0} y$ maps to zero in $(A_1)_{\mathfrak{Q}_1}$, we conclude that y is $(A_1)_{\mathfrak{Q}_1}$-regular. Taking $R_0 \to A_0/yA_0$ and changing base, we know $R_1 \to A_1/yA_1$ is smooth. Since R_1 is Noetherian, we apply (c)⇒(a) to $R_1 \to A_1$, yA_1, and $\mathfrak{Q}_1$, concluding that $R_1 \to (A_1)_{\mathfrak{Q}_1}$ is essentially smooth. Changing base and localizing to $A_{\mathfrak{Q}} = A$, we get, as desired, that $R \to A_{\mathfrak{Q}}$ is essentially smooth. □

Completion of Proof of Theorem 7.3. As mentioned above in the partial proof of Theorem 7.3, we only need to show (3c). By the claim appearing in the partial proof, $F_{j_1}, \ldots, F_{j_r}$ give a minimal generating set for the ideal they generate. So applying Theorem 7.3(3a) and Theorem 7.14, $F_{j_1}, \ldots, F_{j_r}$ is a regular sequence. Thus $B_0 = A/(F_{j_1}, \ldots, F_{j_r})$ has dimension $\dim A - r$, as desired. □

8. Smoothness

Definition 8.1. *Following* [Mat2, §31], *if B is a ring of finite Krull dimension, we say B is* equidimensional *if for all minimal primes $\mathfrak{P} \subseteq B$, $\dim B/\mathfrak{P} = \dim B$. Trivially, domains of finite Krull dimension are equidimensional.*

Proposition 8.2. *Let B be a finite type algebra over a field.*

(1) B is equidimensional if and only if $\dim B_{\mathfrak{M}} = \dim B$ for all maximal ideals $\mathfrak{M} \subseteq B$.

(2) Suppose B is equidimensional. Then for all primes $\mathfrak{P} \subseteq B$, $\dim B = \dim B_{\mathfrak{P}} + \dim B/\mathfrak{P}$.

(3) If $\mathfrak{P} \subseteq B$ is a minimal prime ideal, then there exists a maximal ideal $\mathfrak{M} \subseteq B$ such that $\mathfrak{P}$ is the only minimal prime contained in $\mathfrak{M}$. Further, for such $\mathfrak{M}$, $\dim B_{\mathfrak{M}} = \dim B_{\mathfrak{M}}/\mathfrak{P}B_{\mathfrak{M}} = \dim B/\mathfrak{P}$.

Proof. (2) generalizes Proposition 7.2(2), and is proved by applying it to $B/\mathfrak{P}_0$, for $\mathfrak{P}_0 \subseteq \mathfrak{P}$ a minimal prime such that $\dim B_{\mathfrak{P}} = \dim B_{\mathfrak{P}}/\mathfrak{P}_0 B_{\mathfrak{P}}$.

For (3), let $\mathcal{X}$ be the set of maximal ideals which contain $\mathfrak{P}$, and let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be the minimal primes which are not $\mathfrak{P}$. Since B is finite type over a field, $\mathfrak{P} = \cap_{\mathfrak{M} \in \mathcal{X}} \mathfrak{M}$, by [Mat1, 14.L] or [Mat2, Theorem 5.5]. However, since $\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r \not\subseteq \mathfrak{P}$, there exists $\mathfrak{M} \in \mathcal{X}$ such that $\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r \not\subseteq \mathfrak{M}$. This gives the desired $\mathfrak{M}$ and shows $\dim B_{\mathfrak{M}} = \dim B_{\mathfrak{M}}/\mathfrak{P}B_{\mathfrak{M}}$. Applying (2) to the domain $B/\mathfrak{P}$, we get that $\dim B/\mathfrak{P} = \dim B_{\mathfrak{M}}/\mathfrak{P}B_{\mathfrak{M}} + \dim B/\mathfrak{M} = \dim B_{\mathfrak{M}}/\mathfrak{P}B_{\mathfrak{M}}$.

Finally, for (1), if B is equidimensional, we have the desired equality by (2). Conversely, suppose $\dim B_{\mathfrak{M}} = \dim B$ for all maximal ideals $\mathfrak{M} \subseteq B$, and let $\mathfrak{P} \subseteq B$ be a minimal prime ideal. By (3), we can find a maximal ideal $\mathfrak{M} \subseteq B$ containing $\mathfrak{P}$ such that $\dim B/\mathfrak{P} = \dim B_{\mathfrak{M}} = \dim B$, whence B is equidimensional. □

Lemma 8.3. *Let $R \to B$ be finitely presented, $\mathfrak{p} \subseteq R$ a prime ideal, and $\mathfrak{Q} \subseteq B$ a prime ideal which is maximal with respect to the property $\mathfrak{Q} \cap R = \mathfrak{p}$.*

(1) $\operatorname{tr.deg.}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{Q}) = 0$.

(2) $\dim B_{\mathfrak{Q}}/\mathfrak{p}B_{\mathfrak{Q}} \leq \operatorname{rank}_{\kappa(\mathfrak{Q})} \Omega_{B/R} \otimes_B \kappa(\mathfrak{Q})$, with equality if and only if $\kappa(\mathfrak{p}) \to B_{\mathfrak{Q}}/\mathfrak{p}B_{\mathfrak{Q}}$ is essentially smooth.

(3) If R is a finite type algebra over a field, then $\dim B/\mathfrak{Q} = \dim R/\mathfrak{p}$.

Proof. By maximality, $(B/\mathfrak{Q}) \otimes_R \kappa(\mathfrak{p})$ is the field $\kappa(\mathfrak{Q})$. It is a finite type $\kappa(\mathfrak{p})$-algebra, so to get (1), apply Proposition 7.2(3). For (3), apply it again to get $\dim B/\mathfrak{Q} = \operatorname{tr.deg.}_k \kappa(\mathfrak{Q}) = \operatorname{tr.deg.}_k \kappa(\mathfrak{p}) = \dim R/\mathfrak{p}$, where k is the field over which R and B are finite type. (2) follows from (1) and Theorem 7.3(1a),(1b). □

Theorem 8.4. *For $R \to B$ finitely presented, the following are equivalent.*

(a) $R \to B$ *is smooth.*

(b) $R \to B$ *is flat and* $B = B_1 \times \cdots \times B_t$*, such that each* $\Omega_{B_i/R}$ *is projective of rank* n_i*, and for every prime ideal* $\mathfrak{p} \subseteq R$*,* $B_i \otimes_R \kappa(\mathfrak{p})$ *is equidimensional of dimension* n_i*.*

(c) $R \to B$ *is flat and for all maximal ideals* $\mathfrak{M} \subseteq B$ *with* $\mathfrak{p} = \mathfrak{M} \cap R$*,* $\dim B_{\mathfrak{M}}/\mathfrak{p}B_{\mathfrak{M}} = \operatorname{rank}_{\kappa(\mathfrak{M})} \Omega_{B/R} \otimes_B \kappa(\mathfrak{M})$.

Proof. For (b)⇒(c), we can verify the equation of (c) separately for each B_i. So assume $\Omega_{B/R}$ is projective of rank n and $B \otimes_R \kappa(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is equidimensional of dimension n for all primes $\mathfrak{p} \subseteq R$. For $\mathfrak{M} \subseteq B$ maximal and $\mathfrak{p} = \mathfrak{M} \cap R$, apply Proposition 8.2(1) to the finite type $\kappa(\mathfrak{p})$-algebra $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ to get $\dim B_{\mathfrak{M}}/\mathfrak{p}B_{\mathfrak{M}} = n$, as desired.

For (c)⇒(a), by Theorem 5.11, we must show for $\mathfrak{M} \subseteq B$ maximal and $\mathfrak{p} = \mathfrak{M} \cap R$ that $\kappa(\mathfrak{p}) \to B_{\mathfrak{M}}/\mathfrak{p}B_{\mathfrak{M}}$ is essentially smooth, which follows from Lemma 8.3(2).

Finally, we show (a)⇒(b), so assume $R \to B$ is smooth. It is flat by Theorem 7.12(1). Since $R \to B$ is a finitely presented map, $\Omega_{B/R}$ is a finitely presented B-module. It is also projective, by Corollary 2.19, so write $B = B_1 \times \cdots \times B_t$ with $\Omega_{B/R} \otimes_B B_i$ projective of constant rank n_i. Since B_i is a localization of B at an element, $R \to B_i$ is smooth and $\Omega_{B_i/R} = \Omega_{B/R} \otimes_B B_i$. So replacing B by B_i, we may assume $\Omega_{B/R}$ is projective of rank n. We must show $B \otimes_R \kappa(\mathfrak{p})$ is equidimensional of dimension n for any prime $\mathfrak{p} \subseteq R$. By Proposition 8.2(1), it suffices to show that if $\mathfrak{Q} \subseteq B$ gives a maximal ideal of the finite type $\kappa(\mathfrak{p})$-algebra $B \otimes_R \kappa(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, we have $\dim B_{\mathfrak{Q}}/\mathfrak{p}B_{\mathfrak{Q}} = n$. This follows from Lemma 8.3(2). □

For maps of finite type rings over a field, condition (b) above can be reformulated using the following proposition, which generalizes [H, III.9.6] and [F, B.2.5].

Proposition 8.5. *Let* $R \to B$ *be a flat map of finite type algebras over a field. Fix* $n \geq 0$*. Then the following are equivalent.*

(a) For all prime ideals $\mathfrak{p} \subseteq R$*,* $B \otimes_R \kappa(\mathfrak{p})$ *is equidimensional of dimension* n*.*

(b) For all maximal ideals $\mathfrak{m} \subseteq R$*,* $B \otimes_R \kappa(\mathfrak{m})$ *is equidimensional of dimension* n*.*

(c) For all prime ideals $\mathfrak{p} \subseteq R$*,* $B/\mathfrak{p}B$ *is equidimensional of dimension* $n + \dim R/\mathfrak{p}$.

(d) For all minimal prime ideals $\mathfrak{p} \subseteq R$*,* $B/\mathfrak{p}B$ *is equidimensional of dimension* $n + \dim R/\mathfrak{p}$.

(e) For all minimal prime ideals $\mathfrak{P} \subseteq B$ *and minimal prime ideals* $\mathfrak{p} \subseteq R$ *such that* $\mathfrak{p} \subseteq \mathfrak{P} \cap R$*,* $\dim B/\mathfrak{P} = n + \dim R/\mathfrak{p}$.

Proof. (a)⇒(b) and (c)⇒(d) are trivial.

For (b)⇒(c), by replacing $R \to B$ by $R/\mathfrak{p} \to B/\mathfrak{p}B$, we may assume that R is a domain, and that we must show B is equidimensional of dimension $n + \dim R$. Let $\mathfrak{M} \subseteq B$ be a maximal ideal and $\mathfrak{m} = \mathfrak{M} \cap R$, which is maximal by Lemma 8.3(3). Since $\mathfrak{m} \subseteq R$ and $\mathfrak{M}/\mathfrak{m}B \subseteq B/\mathfrak{m}B = B \otimes_R \kappa(\mathfrak{m})$ are maximal ideals of equidimensional rings, apply Proposition 8.2(1) to get $\dim R = \dim R_\mathfrak{m}$ and $\dim B_{\mathfrak{M}}/\mathfrak{m}B_{\mathfrak{M}} = \dim B \otimes_R \kappa(\mathfrak{m}) = n$. Applying Proposition 7.2(1), we therefore get $\dim B_{\mathfrak{M}} = \dim R_\mathfrak{m} + \dim B_{\mathfrak{M}}/\mathfrak{m}B_{\mathfrak{M}} = \dim R + n$. Since this holds for all maximal ideals $\mathfrak{M}$, again applying Proposition 8.2(1), we see that B is equidimensional of dimension $\dim R + n$.

For (d)⇒(e), suppose $\mathfrak{P} \subseteq B$ is minimal. Then by Proposition 7.2(1), so is $\mathfrak{P} \cap R$, so in (e), $\mathfrak{p} = \mathfrak{P} \cap R$. Since $\mathfrak{P}$ gives a minimal prime ideal of $B/\mathfrak{p}B$, $\dim B/\mathfrak{P} = n + \dim R/\mathfrak{p}$.

Finally, we show (e)⇒(a). Let $\mathfrak{p} \subseteq R$ be prime. Choose $\mathfrak{p}_0 \subseteq \mathfrak{p}$ a minimal prime ideal. Replacing $R \to B$ by $R/\mathfrak{p}_0 \to B/\mathfrak{p}_0B$ and $\mathfrak{p}$ by $\mathfrak{p}/\mathfrak{p}_0$, we may assume without loss of generality that R is a domain and B is equidimensional of dimension $n + \dim R$. We must show that $B \otimes_R \kappa(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is equidimensional of dimension n. Suppose $\mathfrak{Q} \subseteq B$ gives a maximal prime of $B \otimes_R \kappa(\mathfrak{p})$. Then $B_{\mathfrak{Q}}/\mathfrak{p}B_{\mathfrak{Q}} = \dim B_{\mathfrak{Q}} - \dim R_\mathfrak{p}$ by Proposition 7.2(1), which equals

$(\dim B - \dim B/\mathfrak{Q}) - (\dim R - \dim R/\mathfrak{p})$ by Proposition 8.2(2), and this equals n since $\dim B/\mathfrak{Q} = \dim R/\mathfrak{p}$ by Lemma 8.3(3). Since the dimension of $B \otimes_R \kappa(\mathfrak{p})$ is n at all maximal ideals, by Proposition 8.2(1), $B \otimes_R \kappa(\mathfrak{p})$ is equidimensional of dimension n. □

We now begin to compare various authors' definitions of and terminology for smoothness. Recall that we have followed Swan's definitions [Sw]. As we saw in Section 1, Grothendieck defines formal smoothness [EGA, 0_{IV}.19.3.1] for topological algebras—it is our definition of quasi-smoothness, except in the category of topological rings and continuous ring homomorphisms. Our quasi-smoothness for $R \to B$ corresponds to Grothendieck's formal smoothness when R and B are each given the discrete topology. Grothendieck does not use the term smooth for a ring map.

On the other hand, Matsumura in [Mat1] defines formal smoothness as Grothendieck does, but only for adic ring topologies. Then Matsumura defines a ring map to be smooth if it is formally smooth when each ring is given the discrete topology, i.e. our quasi-smooth. (See [Mat1, 28.B–28.D].) In [Mat2], on the other hand, Matsumura uses the term I-smooth for what in [Mat1] is formally smooth, and there 0-smooth corresponds to our quasi-smooth.

Above variations are in terminology, not substance. Before we look at definitions which are more geometric, we need to adapt our definitions and results to geometric language. Although other results carry over, we will only give those we need for this discussion.

Definition 8.6. *A map of schemes $f : X \to S$ is* locally finitely presented *if there exists an open affine cover $\{X_\alpha\}$ of X and open affine sets $\{S_\alpha\}$ of S such that for all α, $f(X_\alpha) \subseteq S_\alpha$ and $\mathcal{O}_S(S_\alpha) \to \mathcal{O}_X(X_\alpha)$ is a finitely presented map of rings. A map $X \to S$ is* smooth *if it is quasi-smooth and locally finitely presented.*

Let $X \to S$ be a scheme map, $s \in S$, and $K \supseteq k(s)$ be the algebraic closure. The fiber *[resp.* geometric fiber*] over s is $X \times_S \operatorname{Spec} k(s)$ [resp. $X \times_S \operatorname{Spec} K$]; we say it is* smooth *if the natural map to $\operatorname{Spec} k(s)$ [resp. to $\operatorname{Spec} K$] is smooth, and we say it is* regular *if its local rings are all regular local rings. Finally, a finite dimensional scheme is* equidimensional *if every irreducible component has the same dimension.*

Note that if $X \to S$ is locally finitely presented, then for *any* open affine sets $X' \subseteq X$ and $S' \subseteq S$ with $f(X') \subseteq S'$, $\mathcal{O}_S(S') \to \mathcal{O}_X(X')$ is finitely presented [EGA, IV.1.4]. In particular, a map of affine schemes is locally finitely presented iff the corresponding ring map is finitely presented. Also, if $X \to S$ is locally finitely presented, then $\Omega_{X/S}$ is a finitely presented sheaf of $\mathcal{O}_X$-modules, which means smoothness is local by Theorem 4.11 and Corollary 4.13. We also wish to mention that we avoid defining the term non-singular, because there is no uniform definition: it can be defined by regularity [H], smoothness [F], or in other ways [Mu, p. 232]. We now have the language for the following results.

Proposition 8.7. *Let $X \to S$ be a locally finitely presented scheme map.*

(1) $X \to S$ is smooth if and only if X is covered by open affine sets U satisfying all of the following. The image of U is contained in an open affine set of S, and restricted to U, $X \to S$ equals the composition of smooth maps $U \to S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[X_1,\ldots,X_n] \to S$. Algebraically, these maps can be expressed as ring maps $R \to R[X_1,\ldots,X_n] \to R[X_1,\ldots,X_{n+t}]/(F_1,\ldots,F_t) = \mathcal{O}_X(U)$. Also, $\Omega_{\mathcal{O}_X(U)/R[X_i]} = 0$ and $dX_1,\ldots,dX_n$ give a basis of the free $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)/R}$. Finally, the last t rows of the $(n+t)\times t$ Jacobian matrix $\left(\frac{\partial F_j}{\partial X_i}\right)$ is an invertible matrix over $\mathcal{O}_X(U)$.

(2) If $\Omega_{X/S} = 0$, then $X \to S$ is smooth if and only if it is flat.

Proof. Reduce to a map of affine schemes by Theorem 4.11. Then (1) follows from Proposition 6.4, and (2) from Theorem 7.12(2). □

Theorem 8.8. *The following conditions on a locally finitely presented map are equivalent.*

(a) Smooth

(b) Flat with smooth fibers

(c) Flat with smooth geometric fibers

(d) Flat with regular geometric fibers

If these hold, then restricting this map to any connected open set, the fibers and geometric fibers are both equidimensional, and both have dimension equal to the rank of the sheaf of differentials on the open set.

Proof. Reduce to a map of affine schemes by Theorem 4.11. Use Theorem 7.12(1) for (a)⇔(b), Proposition 4.6(2) for (c)⇒(b) ((b)⇒(c) being trivial), and Proposition 7.6(2) for (c)⇔(d). In the final statement, rank is well defined and constant by Corollary 2.19 and connectedness. Then apply Theorem 8.4 to the fiber map or the geometric fiber map, since rank of the sheaf of differentials is stable under base change. □

Let us now look at other definitions of smoothness of a map of schemes. As we saw in Section 1, Grothendieck's definition for a map of schemes to be formally smooth exactly corresponds to our definition for quasi-smooth; his definition of a scheme map being smooth is identical to ours, i.e. additionally requiring locally finitely presented. So by omitting "formal," he adds the condition of locally finitely presented [EGA, IV.17.1.1 and IV.17.3.1]. Matsumura, by contrast, omits "formal" to indicate giving each ring the discrete topology, as we saw above; this is consistent with his definition of formally projective [Mat1, 29.B]. It is to avoid this confusion we have used Swan's term quasi-smooth.

In [AK], Altman and Kleiman's definitions assume all schemes are locally Noetherian and maps are locally finite type (which for Noetherian schemes is equivalent to locally finitely presented). They begin by defining an unramified map, showing their definition is equivalent to the sheaf of differentials being zero. Then they define a map to be étale if it is flat and unramified. Using Proposition 8.7(2), this is equivalent to the map being smooth with sheaf of differentials zero. Finally, they define a map to be smooth if it locally factors as an étale map followed by a polynomial map; this is equivalent to smoothness by Proposition 8.7(1). See [AK, (VI, 3.1, 3.3, 4.1) and (VII, 1.1)]; also see Remark 3.5 and Corollary 3.6.

In [Mu], Mumford defines a map $f : X \to S$ of schemes (which, unnecessarily, are assumed to be separated) to be smooth of relative dimension n if for every $x \in X$, we can find schemes making the following diagram commute, where all horizontal maps indicate open immersions, $x \in U$, and evaluating the Jacobian matrix $\left(\frac{\partial F_j}{\partial X_i}\right)$ at the field $k(x)$ gives a matrix of rank t.

$$\begin{tikzcd} X \arrow[d] & U \arrow[l, hook'] \arrow[r, hook] \arrow[d] & \operatorname{Spec} R[X_1, \ldots, X_{n+t}]/(F_1, \ldots, F_t) \arrow[d] \\ S & V \arrow[l, hook'] \arrow[r, hook] & \operatorname{Spec} R \end{tikzcd}$$

Certainly any map satisfying Mumford's definition is locally finitely presented, so by Corollary 4.13, we can check smoothness at each point of X. By the Jacobian Criterion, Theorem 6.1, a map satisfying Mumford's definition is smooth, and by Proposition 8.7(1), a smooth map satisfies Mumford's definition for smooth of relative dimension n, though n may not be the same at all points—it is the rank of the locally free sheaf $\Omega_{X/S}$ at the point x. So for X connected, $X \to S$ is smooth if and only if it satisfies Mumford's definition, and the n which appears is the rank of $\Omega_{X/S}$.

Hartshorne [H] and Fulton [F] have definitions which are similar to one another. In both cases, smoothness is only defined for maps of schemes of finite type over a field (which are, in particular, locally finitely presented). For such a map $f : X \to S$, Hartshorne defines f to be smooth of relative dimension n if (i) f is flat, (ii) for all irreducible components $X' \subseteq X$ and $S' \subseteq S$ such that $f(X') \subseteq S'$, we have $\dim X' = n + \dim S'$, and (iii) for each point $x \in X$, $\mathrm{rank}_{k(x)} \Omega_{X/S} \otimes_{\mathcal{O}_X} k(x) = n$. Fulton, on the other hand, defines f to be smooth of relative dimension n if (i) f is flat, (ii) for all closed irreducible $W \subseteq S$, $f^{-1}(W)$ is equidimensional of dimension $n + \dim W$, and (iii) $\Omega_{X/S}$ is a locally free sheaf of rank n. All these conditions can be checked over affine open sets, as can smoothness, by Theorem 4.11. In both Hartshorne's and Fulton's definitions, because f is assumed flat, condition (ii) is equivalent to requiring all fibers to be equidimensional of dimension n. Indeed, apply Proposition 8.5 (e)$\Leftrightarrow$(c)$\Leftrightarrow$(a). With this reformulation, by Theorem 8.4, these definitions are identical to Mumford's, i.e. smooth with $n = \mathrm{rank}\, \Omega_{X/S}$. Note that either Hartshorne's or Fulton's definition would be improved by substituting for (ii) that all fibers are equidimensional of dimension n, because then the definition would be valid for all locally finitely presented maps.

We end by correcting an error which Mumford makes in Theorem 3′ [Mu, III.10], which states that a finite type map of schemes is smooth of relative dimension n if and only if it is flat and its geometric fibers are disjoint unions of n-dimensional non-singular varieties. To avoid Mumford's definition of non-singularity, which is an unnecessary aside, let us note that at the beginning of the proof of Theorem 3′, Mumford shows that geometric fibers are unions of non-singular varieties if and only if they are smooth. So, using our terminology, Theorem 3′ states that a finite type map of schemes is smooth with sheaf of differentials having rank n if and only if it is flat and its geometric fibers are smooth and equidimensional of dimension n. (We use Theorem 8.4 to get that each connected component, being smooth over a field, is itself equidimensional.)

As stated, the theorem is not true, even on affine schemes. Here is a map which is finite type and flat with smooth geometric fibers, but which is not smooth. Let R^Λ and $\mathfrak{a}$ be as in Example 4.8 and consider $R^\Lambda \to R^\Lambda/\mathfrak{a}$. For all prime ideals $\mathfrak{p} \subseteq R^\Lambda$, $(R^\Lambda/\mathfrak{a})_\mathfrak{p} = R^\Lambda_\mathfrak{p}$ or $(R^\Lambda/\mathfrak{a})_\mathfrak{p} = 0$, so this map is flat and its geometric fibers are algebraically identity maps or maps into the zero ring, either of which is smooth. However, the map, though quasi-smooth (by Example 2.20), is not smooth, because as mentioned earlier in this section, Mumford's definition of smooth, like ours, implies a finitely presented map, but $R^\Lambda \to R^\Lambda/\mathfrak{a}$ fails to be finitely presented since $\mathfrak{a}$ is not finitely generated. This also gives a counterexample to Mumford's Theorem 3, which is the special case of Theorem 3′ with $n = 0$.

The error in Mumford's proof is an unstated assumption that the base scheme is locally Noetherian, an assumption he uses when applying the lemma which appears at the end of the proof. Rather than adding this assumption, a better way to fix Theorem 3′ (and Theorem 3) is to replace the assumption of finite type with locally finitely presented (which, in fact, holds for finite type maps with locally Noetherian base schemes). Indeed, this result would follow from Theorem 8.8.

We should also note that neither Theorem 3′ nor Theorem 3 can be fixed by replacing smooth with quasi-smooth—Example 7.13 gives a finite type map which is quasi-smooth and not flat, and which has module of differentials equal to zero.

REFERENCES

[AK] A. Altman and S. Kleiman, *Introduction to Grothendieck Duality Theory*, Lecture Notes in Mathematics 146, Springer–Verlag, Berlin–New York, 1970.

[AB] M. Auslander and D. A. Buchsbaum, *Groups, rings, modules*, Harper's Series in Modern Mathematics. Harper & Row, Publishers, New York-London, Publishers, New York, 1974; MR **51** #3205.

[B] H. Bass, Big Projective Modules Are Free, *Illinois J. Math.* 7 (1963), 24–31.

[F] W. Fulton, *Intersection Theory*, Springer, 1984.

[EGA] A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique*, I Publ. Math. I.H.E.S. 4 (1960); II, Ibid. 8 (1961); III, Ibid. 11 (1961), 17 (1963); IV, Ibid. 20 (1964), 24 (1965), 28 (1966), 32 (1967).

[H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer–Verlag, Berlin–New York, 1977.

[L] T. Y. Lam, *Serre's Conjecture*, Lecture Notes in Mathematics 635, Springer–Verlag, Berlin–New York, 1978.

[Mat1] H. Matsumura, *Commutative Algebra*, Second edition, Mathematics Lecture Note Series 56, Benjamin/Cummings Publishing Co., Reading, MA 1980.

[Mat2] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Great Britain, 1986.

[Mu] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics 1358, Springer–Verlag, Berlin–Heidelberg, 1988.

[Spn] E. Spanier, *Algebraic Topology*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Company, New York, 1966.

[Spv] M. Spivak, *Calculus on Manifolds*, W. A. Benjamin, Inc., New York, 1965.

[Sw] R. G. Swan, "Néron–Popescu Desingularization," Proceedings of the International Conference on Algebra and Geometry, 1995, International Press Algebra and Geometry Series v. 2, International Press, Boston, MA, 1998. (1995) 135–192.

[Z1] O. Zariski, "Some Results in the Arithmetic Theory of Algebraic Varieties," *American Journal of Mathematics*, Vol. 61, No. 2. (Apr., 1939), pp. 249-294.

[Z2] O. Zariski, "Algebraic Varieties Over Ground Fields of Characteristics Zero," *American Journal of Mathematics*, Vol. 62, No. 1/4. (1940), pp. 187-221.

[Z3] O. Zariski, "The Concept of a Simple Point of an Abstract Algebraic Variety," *Transactions of the American Mathematical Society*, Vol. 62, No. 1. (Jul., 1947), pp. 1-52.

Received: 17.04.2001
Revised: 02.08.2001